Demystifying Hogenauer filters

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Introduction:

In 1981 Eugene B. Hogenauer published a paper on the CIC filter (CIC stands for Cascaded Integrator Comb). He claimed this was an economical class of digital filter. Since then these filters have found extensive use in decimation and interpolation in **multirate systems such as Sigma – Delta Analog to Digital converters**. This paper is an attempt to investigate these types of filters and provide helpful feedback to practicing engineers like ourselves.

CIC Filter structure:

The CIC filter is a cascade of an integrator [$\mathbf{H_I(z) = 1/(1\text{-}z^{-1})}$] and a comb filter [$\mathbf{H_C(z)=(1\text{-}z^{-RM})}$] referenced to the high sampling rate. The comb section operates at a lower sampling rate (f_S/R) than the integrator section. f_S = input high sampling rate. R = sampling rate reduction factor. M is a differential delay typically one or two. This is shown below. Note both filters are one pole filters.

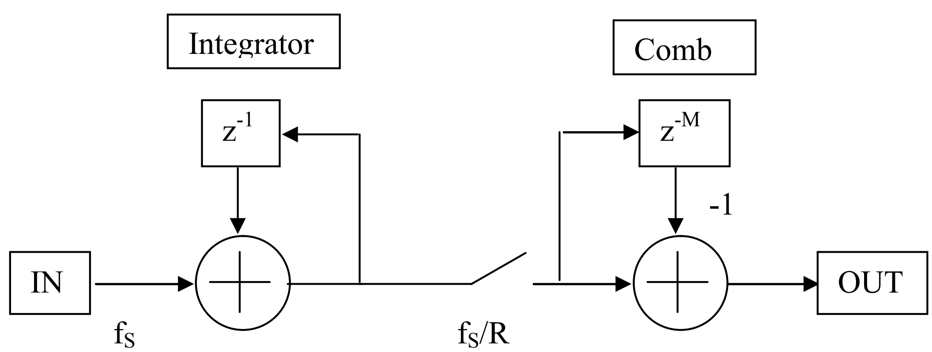

Figure 1.0 A CIC filter with single integrator and comb section.

A few notes are in order at this stage in keeping with the elaboration of the filters operation

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First where do the z – transform based system functions come from?

The digital integrator has a time based equation of:

$y(n) = x(n) + y(n-1)$ Eqn (1).

Or, [the output sample at the nth clock is the sum of the input sample and previous output sample.]

$y(n) - y(n-1) = x(n)$ Eqn (2).

Or,

Taking z transforms on both sides (where the definition of the z transform is: $X(z) = \sum x(n)z^{-k}$. Where the summation is $k = -\infty$ to ∞.) The variable z is complex.

Thus if the z transform is taken on both sides of Eqn (2) we get:

$\mathbf{Y(z) - Y(z)z^{-1} = X(z)}$ Eqn (3).

Note the signal y(n-1) simply means that the signal is a digital sample delayed by one clock and so on. The z transform of delayed signals have a negative integer exponent on the z variable as shown above.

Thus it is obvious that the transfer function is:

$\mathbf{H_I(z) = Y(z)/X(z) = 1/(1-z^{-1})}$ Eqn (4).

Similarly the time based equation for the comb is:

$y(n) = x(n) - x(n-M)$ Eqn (5)

Which leads to:

$Y(z) = X(z) - X(z)z^{-M}$ Eqn (6)

Or,

$\mathbf{H_C(z) = Y(z)/X(z) = 1- z^{-M}}$ Eqn (7)

Generally single sections of the CIC filter do not have good attenuation properties so a cascade of single sections are used with a down sampler between the integrator cascade and the comb cascade as shown below:

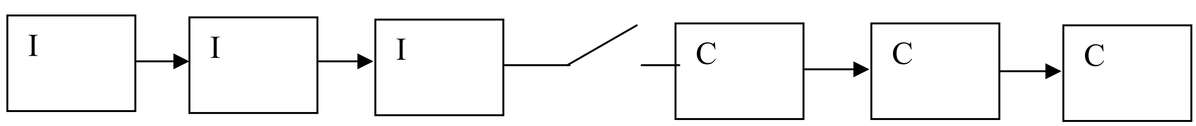

Figure 2: 3 section cascade

This concept can be generalized to N sections. Then the system transfer function becomes:

$$\mathbf{H(z) = H_I(z).H_C(z) = [(1-z^{-RM})/(1-z^{-1})]^N} \qquad \text{Eqn (8)}$$

This is referenced to the high sampling rate.

However, if the single section comb filter transfer function is expanded by polynomial division (i.e divide $1 - z^{-RM}$ by $1 - z^{-1}$) the result will be:

$\boldsymbol{\Sigma}\ \mathbf{z^{-k}}$**.** *The summation is over k from 0 to RM-1.*

$$[\boldsymbol{\Sigma}\ \mathbf{z^{-k}}]^N \ \textbf{for N sections.} \qquad \text{Eqn(8.1)}$$

This summation is the same as the system function for a FIR filter. Therefore a N section CIC filter is functionally equivalent to a cascade of N FIR filters!

A way to do this is use a N cascade of RM storage registers (FF's) and one accumulator per section.

The advantages and disadvantages stated by Hogenauer in his original paper for these filters are repeated below:

Advantages and disadvantages:

Advantages:

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1.0 No multipliers are required.
2.0 No storage is required.
3.0 Intermediate storage is reduced by integrating at a high sampling rate and comb filtering at a low sampling rate compared to the equivalent implementation using cascaded uniform FIR filters.
4.0 The structure of the CIC filter is very "regular" consisting of two basic building blocks.
5.0 Little external control or complicated local timing is required.
6.0 The same filter design can easily be used for a wide range of rate change factors, R, with the addition of a scaling circuit and minimal changes to the filter timing.

"

Disadvantages:

"

1.0 Register widths can become large for large rate change factors.

2.0 The frequency response is fully determined by only three integer factors (R, M and N), resulting in a limited range of filter characteristics.

Typical usage of CIC filters is in " areas where high sampling rates make multipliers an uneconomical choice and areas where large rate change factors would require large amounts of coefficient storage or fast impulse response generation."

Frequency response:

CIC filters are essentially low pass filters. Their frequency and power response can be derived as follows:

Evaluate the transfer function H(z) at:

$$z = e^{j(2\pi f/R)} \qquad \text{Eqn 9.0}$$

Thus:

$$H(f) = \frac{1 - e^{-(j2\pi f/R)RM}}{1 - e^{-(j2\pi f/R)}} \qquad \text{Eqn 10.0}$$

$$\frac{1 - e^{-(j2\pi fM)}}{1 - e^{-(j2\pi f/R)}} \qquad \text{Eqn 11.0}$$

Let us now use the numerator to first simplify the above equation in a series of steps shown below:

$$1 - e^{-(j2\pi fM)} = e^{-(j2\pi fM)/2}\, e^{(j2\pi fM)/2} - e^{-(j2\pi fM)} \qquad \text{Eqn 12.0}$$

$$= e^{-(j2\pi fM)/2} \left(e^{(j2\pi fM)/2} - e^{-(j2\pi fM)} / e^{-(j2\pi fM)/2)} \right)$$

Eqn 13.0

$$= e^{-(j2\pi fM)/2} \left(e^{(j2\pi fM)/2} - e^{-(j2\pi fM)/2} \right) \qquad \text{Eqn 14.0}$$

$$= e^{-(j2\pi fM)/2} \left(\mathrm{Cos}\,\pi fM + j\mathrm{Sin}\,\pi fM - \mathrm{Cos}\,\pi fM + j\mathrm{Sin}\,\pi fM \right) \qquad \text{Eqn 15.0}$$

$$= e^{-(j2\pi fM)/2} \left(2j\mathrm{Sin}\ \pi fM \right) \qquad \text{Eqn 16.0}$$

Similarly we get for the denominator:

$$= e^{-(j2\pi f/R)/2} \left(2j\mathrm{Sin}\ \pi f/R \right) \qquad \text{Eqn 17.0}$$

Combining the two we get;

$$H(f) = \left[\frac{\mathrm{Sin}\ \pi fM}{\mathrm{Sin}\ \pi f/R} e^{-(j\pi f/R(RM-1)} \right]^N$$

Eqn 18.0

Again using Euler's identity:

$$= \mathbf{S} \left(\mathrm{Cos}\ \pi f/R(RM-1) - j\ \mathrm{Sin}\ \pi f/R(RM-1)\right)$$
$$= \mathbf{S}\mathrm{Cos}\ \pi f/R(RM-1) - j\ \mathbf{S}\mathrm{Sin}\ \pi f/R(RM-1)) \qquad \text{Eqn 19.0}$$

Where $\mathbf{S} =$ Sin πfM/ Sin πf/R

The amplitude of this response is:

$$= \{[(S^2\mathrm{Cos}^2\ \pi f/R(RM-1) + S^2\mathrm{Sin}^2 \pi f/R(RM-1))]^{1/2}\}^N$$

$$= S^N$$

$$\boxed{\mathbf{|\ H(f)\ | = [Sin\ \pi fM/\ Sin\ \pi f/R]^N}} \qquad \text{Eqn 20.0}$$

The power response is: (Square the term)

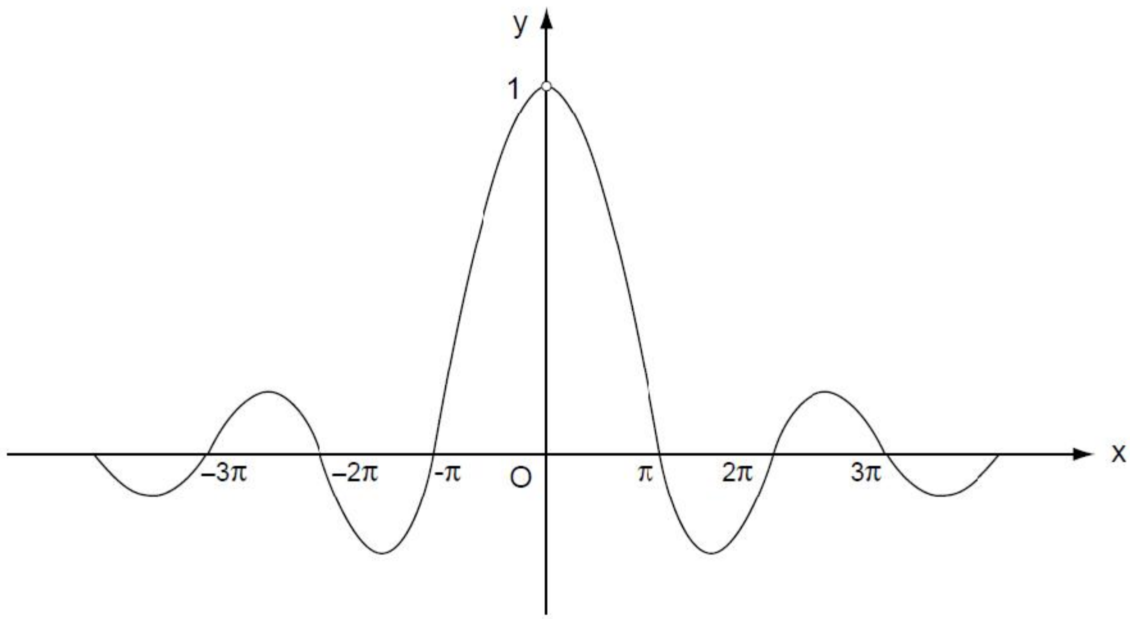
$$\mathbf{|P(f)| = [Sin\ \pi fM/\ Sin\ \pi f/R]^{2N}} \qquad \text{Eqn 21.0}$$

For large R the term, Sin πf/R = πf/R (over a limited frequency range) so:

$$\mathbf{|P(f)| = [RM(Sin\ \pi fM/\ \pi fM)]^{2N}} \qquad \text{Eqn 22.0}$$

This is an approximation that can be used for many practical designs and has an error less than 1 dB for $RM \geq 10$, $7 \geq N \geq 1$ and $255/256M \geq f \geq 0$.

As can be seen the function **Sin πfM/ πfM** is a Sinc function. i.e a Sinx/x function. The graph of this function is shown below:



The function is defined for all values of x except 0; but we also know that as x gets smaller and smaller, the ratio sin x./x—provided x is measured in radians—
tends to 1. We can simply *define* the value of sin0/0 to be 1,
and this definition will assure the continuity of the function near x = 0.

It is symmetric about the y-axis; that is, f(−x) = f(x) for all values of x (in the language of algebra, f (x.) is an *even function*.) the graph of sin x./x represents *damped* oscillations whose amplitude steadily decreases as x increases.

As can be seen from the above graph, the nulls of the function exist at multiples of π.

Therefore using the argument of the power response above, the nulls of the response exist at multiples of f=1/M. We can use M as a design parameter to control where
the nulls of the power response will occur.

The figures below represent the frequency response of a Hogenauer filter.

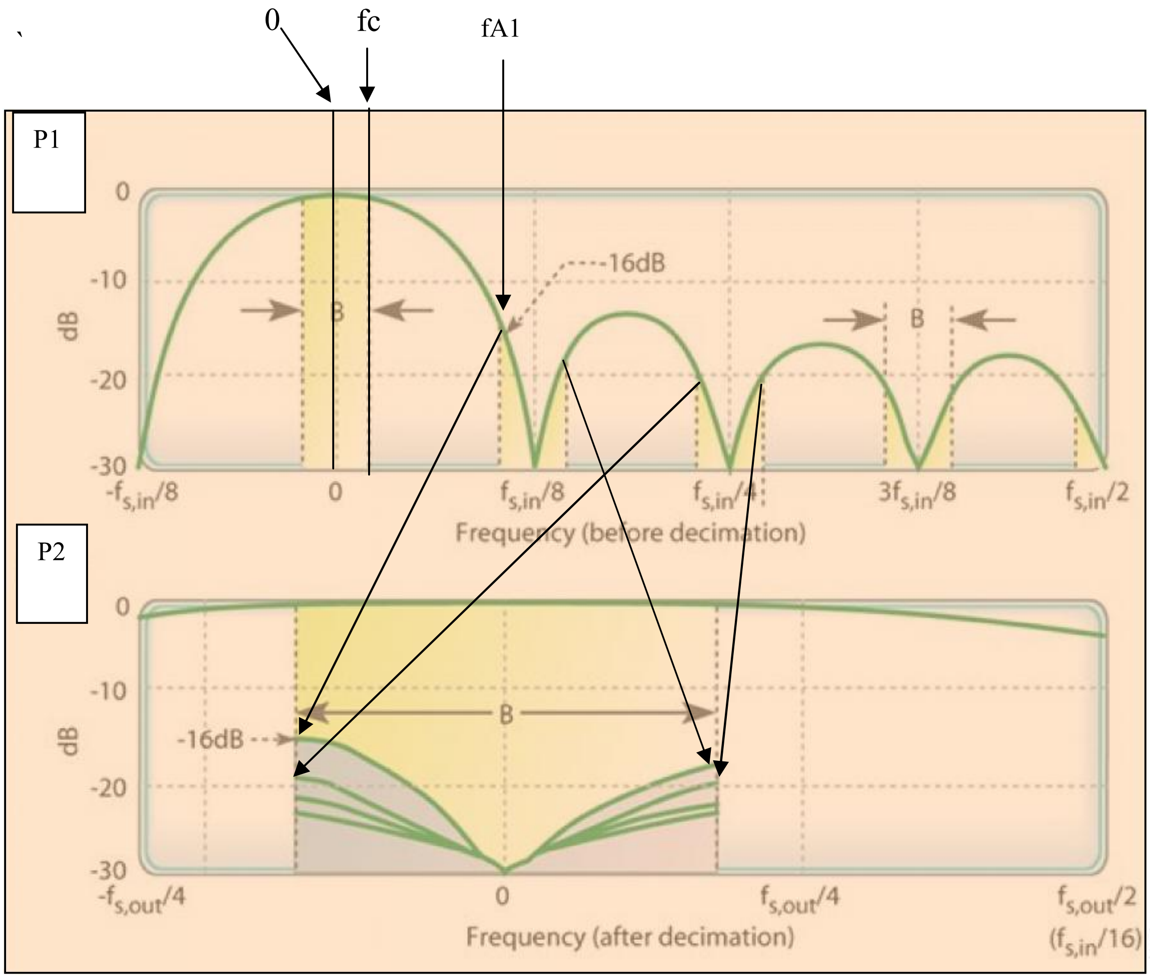


Figure 3.0 Frequency response of a Hogenauer filter

The arrows indicate the folding or aliasing of the areas around the sampling frequency multiples that are B wide. In the figure above the worst case folding occurs for the first multiple of the sampling frequency and is 16 dB below the maximum (0dB) response at fs=0.

In practical design problems, the aliasing errors can be characterized by the maximum error over all aliasing bands. If $fc=B/2 \leq 1/(2M)$ then this maximum error occurs at the lower edge of the first aliasing band at

$$fA1 = 1 - fc \qquad \text{Eqn 23.0}$$

Hogenauer presented tables, reproduced below which assist in determining the tradeoffs between bandwidths, passband attenuation and aliasing error. It is assumed that R, the rate change factor is large so the approximation in equation 22 can be applied.

In the tables presented the attenuations are relative to maximum filter response at f = 0.

Table 1.0

Relative bandwidth – differential delta product (Mfc)	Passband attenuation at fc (dB) as a function of number of stages (N). *					
	1	2	3	4	5	6
1/128	0.0	0.0	0.0	0.0	0.0	0.01
1/64	0.0	0.01	0.01	0.01	0.02	0.02
1/32	0.01	0.03	0.04	0.06	0.07	0.08
1/16	0.06	0.11	0.17	0.22	0.28	0.34
1/8	0.22	0.45	0.67	0.90	1.12	1.35
1/4	0.91	1.82	2.74	3.65	4.56	5.47

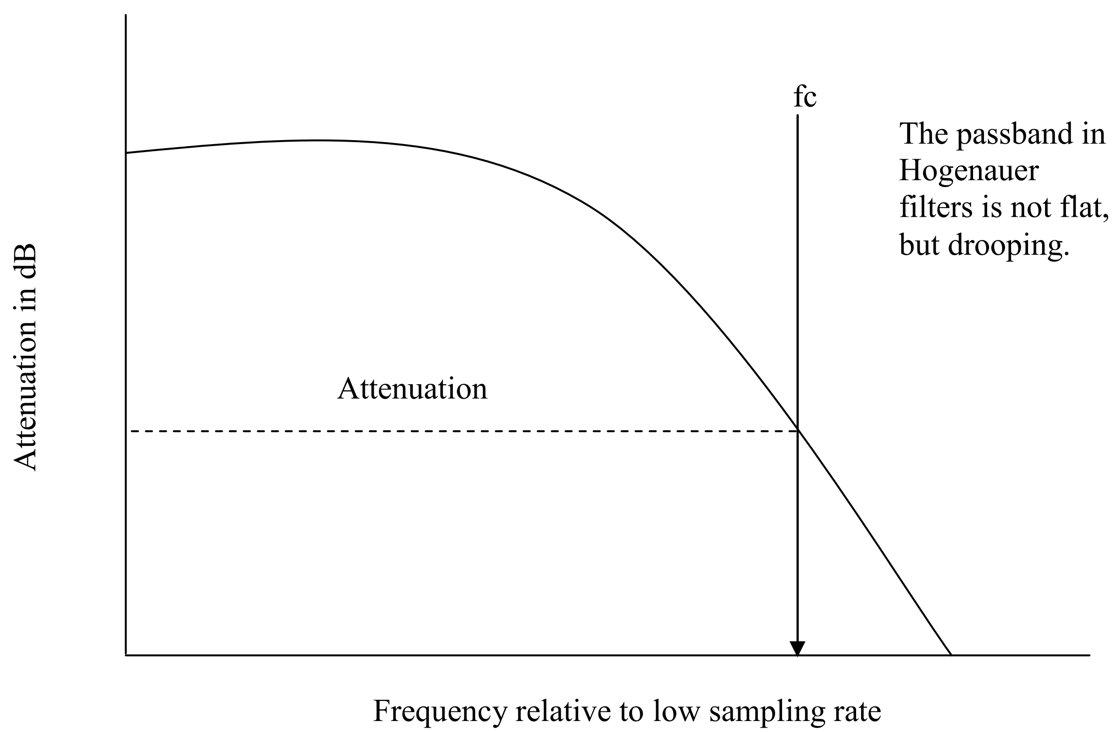


Figure 4.0. Expanded view of frequency response in the vicinity of fc

The other factor that comes into play for these filters is M, the differential delay. In a practical sense factors of M greater 2 seem to be of less value.

Table 2.0

M	fc	Aliasing attenuation at fA1 (dB) as a function of the number of stages N.					
		1	2	3	4	5	6
1	1/128	42.1	84.2	126.2	168.3	210.4	252.5
1	1/64	36.0	72.0	108.0	144.0	180.0	215.9
1	1/32	29.8	59.7	89.5	119.4	149.2	179.0
1	1/16	23.6	47.2	70.7	94.3	117.9	141.5
1	1/8	17.1	34.3	51.4	68.5	85.6	102.8
1	1/4	10.5	20.9	31.4	41.8	52.3	62.7
2	1/256	48.1	96.3	144.4	192.5	240.7	288.8
2	1/128	42.1	84.2	126.2	168.3	210.4	252.5
2	1/64	36.0	72.0	108.0	144.0	180.0	216.0
2	1/32	29.9	59.8	89.6	119.5	149.4	179.3
2	1/16	23.7	47.5	71.2	95.0	118.7	142.5
2	1/8	17.8	35.6	53.4	71.3	89.1	106.9

Conclusions and discussions:

The Hogenauer or CIC filter is a convenient filter for decimation or interpolation of digital signals, for example, in oversampled (OSR) A/Ds (sigma delta A/D). The shape of the response is such that a post filter is usually used to compensate for the droop in the passband response. The tables given above can be used in the design of these filters. A companion paper to this paper focuses on the design of decimation filters as part of a series on OSR A/D converters.