Two New Equations for the Design of Filters*

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1. Introduction

THERE are some situations where a selective circuit that is equivalent to an inverse arm, i.e. a constant-K-configuration filter, is to be designed, *and the unloaded Q of the elements to be used is sufficiently high for them to be considered "nondissipative."* This paper presents two equations that for the nondissipative case, specify the exact element values required for the filter to produce that attenuation shape having the highest possible rate of cutoff, i.e., the Chebishev attenuation shape.

2. Examples of Nondissipative Equivalent Inverse-Arm Filters and the Three Circuit Constants That Must Be Correctly Adjusted

Because so many of the selective circuits now being used, or designed, seem physically so different from the basic inverse-arm configurations, many engineers new to the field do not realize that the design equations for the constant-K configuration can be applied.

It thus seems worthwhile calling attention to a few of these equivalent inverse-arm filters to stress the wide applicability of the two design equations to be presented.

It will be noticed that with one exception, the band-pass examples are from the ultra-high-frequency and microwave regions, because it is mainly in these regions that the ratio of unloaded Q to fractional midfrequency $Q_0/[f_0/(\mathrm{BW})]$ is high enough for the elements to be considered nondissipative.

Figure 1A shows a common direct-coupled waveguide band-pass filter using four resonators; in another language it would be called a quadruple-tuned band-pass filter. The equivalence of this to the fundamental constant-K configuration (either band-pass or low-pass) has been excellently described in W. W. Mumford's paper.[1]

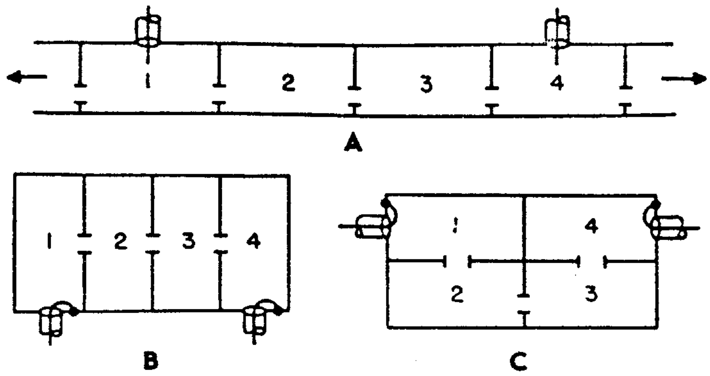

Figure 1—Three ways of arranging 4 waveguide resonators to produce a quadruple-tuned band-pass filter. When small-percentage bandwidths are used, these filters can be designed from the equations for the simple constant-k-configuration low-pass filter of Figure 5.

In the "language" used in this present paper, the design information that the engineer must possess (and which is required for all equivalent constant-K-configuration filters) is:

A. The required coefficient of coupling $K_{r(r+1)}$ between adjacent resonators. This fixes the size of the opening that must be made in the wall between adjacent resonators, and as is well known, this opening can take the form of a slot parallel to the electric-field vector, which will give the equivalent of mutual-inductance coupling between resonators; a slot perpendicular to the electric-field vector, which will give the equivalent of "low-side" capacitive coupling between resonators; a post parallel to the electric-field vector, which will give the equivalent of self-inductance coupling between resonators; or, in general, any kind of opening that will allow some of the electric and/or magnetic field of one resonator to enter the adjacent resonator.

B. The required resonant frequency (f_0) of each resonator. This fixes the distance between the walls of each resonator. As is well known, the coefficient-of-coupling mechanism must be correctly considered a part of each resonator to which it is connected;

* A condensed version of this paper appeared in *Convention Record of the 1953 I.R.E. National Convention*, Part 5—Circuit Theory, pages 44–47. This is the full version of the paper as presented at the National Convention of the Institute of Radio Engineers in New York, New York, on March 23, 1953.

[1] W. W. Mumford, "Maximally-flat Filters in Waveguide," *Bell System Technical Journal*, volume 27, pages 684–713; October, 1948.

otherwise the pass-band midfrequency will not coincide with the resonant frequency.

C. The required singly loaded Q (Q_1) of the first resonator (produced by correctly coupling the generator to this first resonator); and the required singly loaded Q (Q_n) of the last resonator (produced by correctly coupling the load to this last resonator).

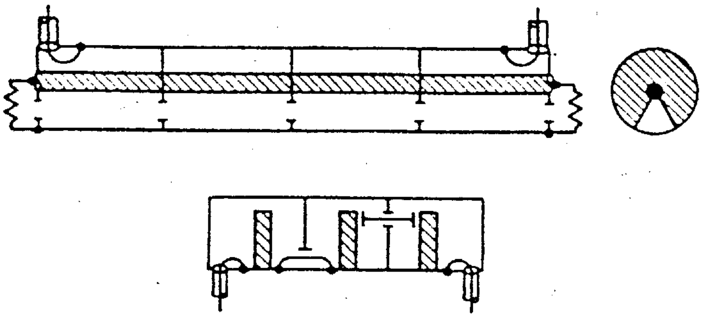

Figure 2—Two of the many ways of using coaxial resonators to produce a small-percentage-band-pass filter. Here, also, the low-pass-design information is applicable.

If a terminated waveguide is used on each side of the filter, this then fixes the size of the opening in the first and last wall of the structure of Figure 1A; or, if desired, these first and last walls can be completely closed off and, as Figure 1A attempts to show, the generator and load can be capacitively coupled to the first and last resonators by probes (or magnetically coupled by loops). Whatever the method used, this generator and load coupling must be adjusted until the first and last resonators, respectively, have the required singly loaded Q_1 and Q_n.

The above three well-known circuit constants have been discussed in a previous paper,[2] and methods of measuring and adjusting them have also been presented.[3]

Continuing with some other examples of equivalent constant-K structures, Figures 1B and 1C show that by discarding the waveguide concept in favor of the coupled-resonator concept, additional useful, and different-looking, filters can be built with the same four resonators. Figure 1B shows the four resonators of Figure 1A rotated by 90 degrees and placed together in such a way that the openings between adjacent resonators produce the equivalent of "high-side" capacitive coupling. Figure 1C shows the same four resonators arranged in yet another physical configuration that will still produce the same small-percentage bandwidth filtering action: there is equivalent "high-side" capacitive coupling between resonators *1* and *2*, mutual-inductance coupling (due to a vertical slot) between resonators *2* and *3*, high-side capacitive between resonators *3* and *4*; the generator sets Q_1 by inductive coupling to the first resonator and the load sets Q_4 by inductive coupling to the last resonator.

Figure 2 is included to stress the fact that the "different-looking" filters produced by using coaxial resonators are also equivalent to constant-K-configuration filters insofar as band-pass response and required circuit constants are concerned.

Figure 3 shows a triple-tuned band-pass filter that, while in no way physically resembling the classical inverse-arm structure, is still described by exactly the same design constants as the inverse-arm structure. It is the spherical resonator that is so designed that three of its resonant modes occur at the same frequency, i.e., are degenerate. The two screws shown project into the cavity and correctly adjust K_{12} (the coefficient of coupling between the first resonance and the second resonance), and K_{23} (the coefficient of coupling between the second resonance and the third resonance). The opening on the left is of the proper size to allow the terminated waveguide shown to load properly the first resonance, i.e., to set Q_1; and the opening on the right allows the terminated waveguide shown there to load

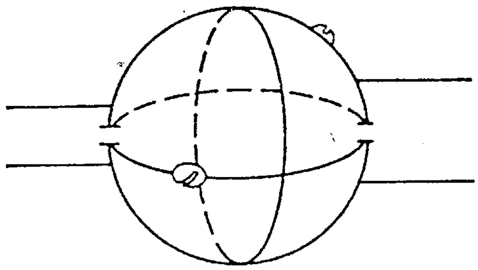

Figure 3—A 3-mode single-cavity microwave filter. While bearing no physical resemblance whatsoever to the classical inverse-arm filter, the low-pass-design information is also applicable to this structure, when small-percentage bandwidths are required.

[2] M. Dishal, "Design of Dissipative Band-Pass Filters Producing Desired Exact Amplitude—Frequency Characteristics," *Electrical Communication*, volume 27, pages 56–81; March, 1950: also, *Proceedings of the I.R.E.*, volume 37, pages 1050–1069; September, 1949.

[3] M. Dishal, "Alignment and Adjustment of Synchronously Tuned Multiple-Resonant-Circuit Filters," *Electrical Communication*, volume 29, pages 154–164; June 1952: Addendum, volume 29, page 292; December, 1952: also, *Proceedings of the I.R.E.*, volume 39, pages 1448–1455; November, 1951.

properly the last resonance, i.e., to set Q_3. Finally, Figure 4 shows a three-resonator filter using mechanical resonators for the filter elements. Here, the coefficient of couplings K_{12} and K_{23} are set by the material, diameter, and "tap" point used for the quarter-wavelength-long (approximately) thin rods that connect two adjacent resonators. Q_1 of the first resonator is correctly set by the thin low-Q resonant rod connected to the first resonator, and the last resonator is similarly correctly loaded by the low-Q rod connected to it. The coils, by magnetostrictive action, convert the electric energy to mechanical energy and then vice versa; because of the unfortunately poor coupling produced by this phenomenon, there is usually negligible electrical loading coupled into the first and last resonators.

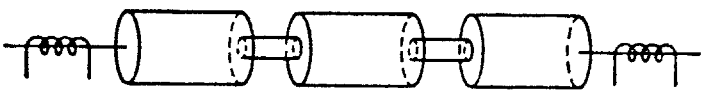

Figure 4—A triple-tuned mechanical filter made with half-wave slugs and quarter-wave coupling necks. The low-pass-design equations apply here also.

There are many other examples of filters that at first glance do not resemble the basic inverse-arm configuration, but that actually are equivalent to it; and in all of these many filters, the design engineer must know the required numerical value for all the coefficients of couplings in the structure; the required numerical value of the singly loaded Q of the first resonator and that of the last resonator; and the proper element values or physical lengths to produce the proper midfrequency (or design information exactly equivalent to these three quantities).

3. *"Incorrect" Coefficients of Coupling and End Q's Called for by Classical Filter Theory*

It will now be assumed that the reader realizes that, within the small-percentage-band-pass approximation, the transfer equation for the circuits of Figures 1 to 4 in terms of the frequency variable

$$\left(\frac{f}{f_0} - \frac{f_0}{f}\right) = \frac{f_2 - f_1}{(f_2 f_1)^{1/2}} = \left(\frac{\mathrm{BW}}{f_0}\right)$$

is identical in form to that of the low-pass ladder of Figure 5, when the frequency variable for the latter is radian frequency ω.

To clarify the meaning of coefficients of coupling and end Q's as applied to the low-pass inverse-arm ladder, classical filter theory will be applied to the ladder of Figure 5 to obtain the design values called for by this theory. It will be recalled that the two basic facts of this theory are as given in (1) and (2).

The full series-arm reactance and the full shunt-arm reactance must be so related that (1) is true at the desired cutoff frequency ω_c.

$$\frac{Z_{\mathrm{series}}}{4Z_{\mathrm{shunt}}} = -1. \qquad (1)$$

The impedance Z_0, which must be used to terminate the ladder and which unfortunately cannot be physically realized, is given by (2).

$$Z_0 = (Z_{\mathrm{series}} Z_{\mathrm{shunt}})^{1/2} (1 + Z_{\mathrm{series}}/4Z_{\mathrm{shunt}})^{1/2}. \qquad (2)$$

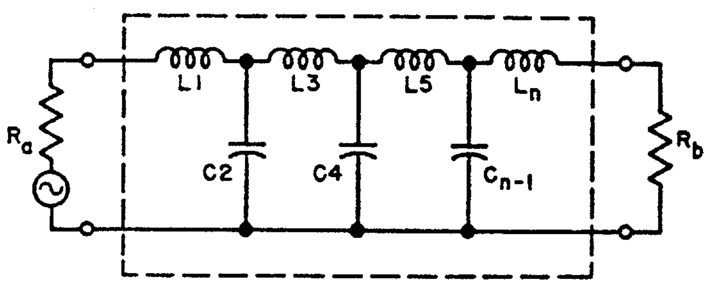

Figure 5—The inverse-arm low-pass ladder whose design equations apply to the small-percentage band-pass filters of Figures 1 to 4. The unloaded Q's of the elements are infinite.

From (1), we obtain (1A),

$$\left.\begin{aligned} 1/(LC_{\mathrm{internal}})^{1/2} &= 0.50\omega_c, \\ \frac{1}{(L_1C_1)^{1/2}} = \frac{1}{(C_{(n-1)}L_n)^{1/2}} &= 0.707\omega_c, \end{aligned}\right\} (1A)$$

and from (2), we obtain

$$\frac{Z_0/L_{1,n}}{\omega_c} = \left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]^{1/2}. \qquad (2A)$$

The required resonant radian frequencies given by (1A) for the adjacent arms of the low-pass ladder of Figure 5 are exactly equivalent to the well-known coefficients of coupling of the small-percentage-band-pass networks of Figures 1 to 4, when the frequency variable ω and BW/f_0 are used, respectively. Thus by (1A), classical filter theory requires that all the internal coefficients of coupling be made equal to 0.50 times the "cutoff" frequency variable, and that the coefficient of coupling between the first and second elements and between the next to the last and last elements be made 0.707 times the cutoff-frequency variable. Simultaneously, (2A) must be satisfied, and this is unfortunately impossible for it demands that at *zero frequency* $R_0/L_{1,n}$ —which is exactly equivalent to the well-known decrement $(1/Q)$ of the small-percentage-band-width networks—must equal 1.0 times the cutoff-frequency variable but then over the pass and attenuation band, the termination must vary in the way specified by the right-hand side of (2A); this is impossible to achieve.

If it were possible to obtain this termination and the above values of coefficient of coupling were used, then the response obtained would be that shown in Figure 6A.

The procedure usually resorted to is to approximate Z_0 by a resistance equal to the zero-frequency value of Z_0, and thus the classical filter is terminated in a fixed resistance of value $R/L_{1,n} = 1.0\omega_c$, i.e. an input and output decrement of 1.0 times the frequency variable is used. The coefficients of coupling between all internal elements are still maintained equal to 0.50 times the frequency variable, and the coefficients of coupling between the first and second elements and next to last and last elements are kept at 0.707 times the frequency variable. When this design is used, it is now well known that the type of response obtained is that[4] shown in Figure 6B. The increasing peak-to-valley ratio in the pass-band makes this type of response objectionable if it is necessary to use a large number of elements to obtain a high rate of cutoff.

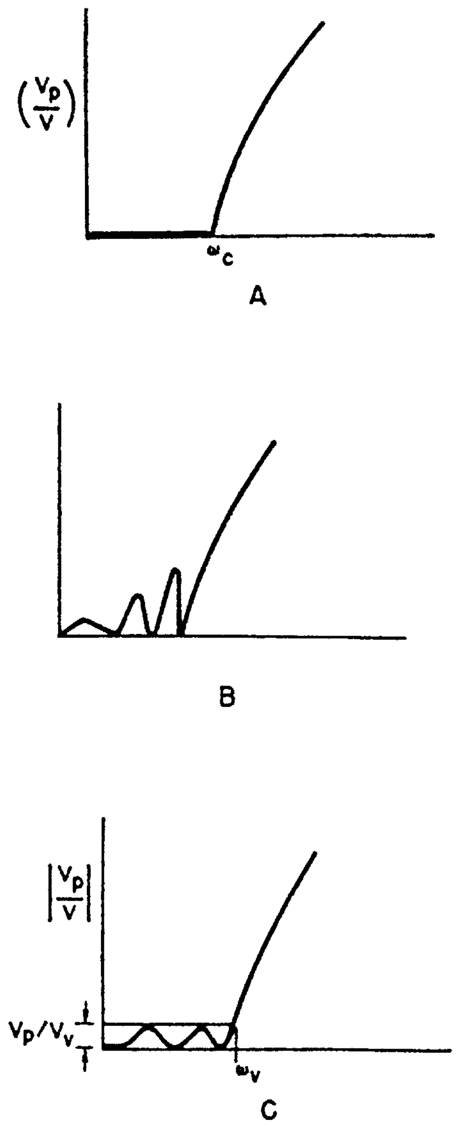

Figure 6—*A* shows the physically unrealizable attenuation shape of a Z_0-terminated constant-*k* filter. $(V_p/V) = \exp(n-1)\cosh^{-1}(\omega/\omega_c)$. When physical resistors are used instead of the unrealizable Z_0, the attenuation shape of *B* is obtained. *C* shows the optimum Chebishev shape, which can be obtained when the element values are correctly modified from the constant-*k* values.

$$|V_p/V|^2 = 1 + [(V_p/V_v)^2 - 1]\cosh^2[n\cosh^{-1}(\omega/\omega_c)].$$

4. *Optimum Constant-K-Configuration Attenuation Shape of Modern Filter Theory*

When there are stringent requirements on the pass-band tolerance and rate of cutoff of a filter, the approximate design procedure of image-parameter theory is usually discarded and the much-more-exact insertion-loss design procedure is used. It is interesting to note that although this procedure was originated more than 15 years ago,[5–7] most practicing engineers are still not familiar with it.

By a series of steps, the history of which is not very clear, it was realized that the nonoptimum

[4] G. L. Ragan, "Microwave Transmission Circuits," McGraw-Hill Book Company, New York, New York; 1948: chapter 9 by R. M. Fano and A. W. Lawson; pages 576 and 577.

[5] E. L. Norton, "Constant Resistance Networks with Applications to Filter Groups," *Bell System Technical Journal*, volume 16, pages 178–193; April, 1937.

[6] S. Darlington, "Synthesis of Reactance 4-Poles," *Journal of Mathematics and Physics*, volume 18, pages 257–353; September, 1939.

[7] E. L. Norton, United States Patent 1 788 538; January, 1931.

response of Figure 6B could be modified so that the ripples in the pass band were all of equal level. The obvious application of the known method of approximating a constant by means of Chebishev polynomials then led to the following basically important equation for the optimum response shown in Figure 6C. For the band-pass case, BW/f_0 would simply be used instead of ω.

$$\left|\frac{V_p}{V}\right|^2 = 1 + \left[\left|\frac{V_p}{V_v}\right|^2 - 1\right] \cosh^2\left(n \cosh^{-1} \frac{\omega}{\omega_v}\right). \tag{3}$$

Where amplitude filtering only is concerned, i.e. no phase- or time-response considerations are involved, the attenuation shape of Figure 6C is optimum in the sense that for a given allowable ripple in the pass band, (3) produces the maximum possible rate of cutoff for a given number of elements.

In (3), n is the number of arms in the ladder of Figure 5 and ω_v is the radian frequency at that point on the skirt where the attenuation is the same as the peak-to-valley ratio.

When the ripple in the pass band is made to become zero decibels, then by correctly approaching this limit, (3) becomes (4).

$$\left|\frac{V_p}{V}\right|^2 = 1 + \left(\frac{\omega}{\omega_{3db}}\right)^{2n}, \tag{4}$$

where ω_{3db} is the radian frequency at that point on the skirt that is 3 decibels down from the peak response.

The problem now is to apply the procedures of modern network theory to the attenuation shapes (3) and (4) and to synthesize the network and element values that will produce this optimum filtering shape.

5. *Two Basic Synthesis Procedures of Modern Network Theory*

Given the equation for a desired attenuation shape (it must be a rational function, of course), there are at least two basic procedures for synthesizing a corresponding network. For want of a better name, the first procedure will be called the direct method and as far as this writer is aware, Norton was the first to describe this method.[7] The second method is properly called the Darlington method and is basically described in reference 6.

5.1 Direct Method

Step 1. Pick a network configuration that is known (somehow) to be capable of producing the desired attenuation shape.

Step 2. By means of Kirchhoff's laws, write the complex equation for the network response to be synthesized. For the low-pass ladder, this equation will be in the form of the ratio of two polynomials in $j\omega$ of highest power n, and with constant coefficients made up of complicated combinations of the L, C, and R elements of the arms of the network.

Step 3. Solve the desired attenuation-shape equation (e.g. (3) or (4)) for its $2n$ complex zeros; multiply these ω zeros by j to make them functions of $j\omega$, and then combine the n left-half-plane zeros to obtain the complex-numerator polynomial in $j\omega$ of exactly the same form as that obtained in Step *2* but with numerical coefficients. Next, solve for the $j\omega$ attenuation infinities and use them to form the denominator polynomial.

Step 4. Compare the equations of Step *2* and Step *3* and equate the coefficients of identical powers of $j\omega$. This will result in n simultaneous equations that must be simultaneously solved for the n unknown element values.

Note: This method was described in detail in reference 1 at a time when the author did not know of the existence of references 5 and 7.

5.2 Darlington Method for a Uniformly Dissipative Filter

Step 1. Solve the desired attenuation-shape equation ((3) or (4)) for its $2n$ complex zeros. Multiply these ω/ω_v zeros by j so that they are functions of $j(\omega/\omega_v)$.

Step 2. Pick the n left-half-plane zeros and reduce the magnitude of the real component of these zeros by an amount equal to the normalized decrement of each arm, i.e. by an amount $(R/L)/\omega_v = (G/C)/\omega_v$ for the low-pass ladder.

Step 3. Use these modified n left-half-plane zeros to form a complex-numerator polynomial in $j(\omega/\omega_v)$ having numerical coefficients.

Solve the desired attenuation-shape equation for its $j(\omega/\omega_v)$ infinities and use these infinities to form a denominator polynomial in $j(\omega/\omega_v)$.

Our modified complex-shape equation will now be in the form of (5).

$$\frac{V_p}{V} = \frac{1}{|\Delta|_{min}} \left\{ \frac{\left(j\frac{\omega}{\omega_v}\right)^n + U_{n-1}\left(j\frac{\omega}{\omega_v}\right)^{n-1} + U_{n-2}\left(j\frac{\omega}{\omega_v}\right)^{n-2} + \ldots + U_0}{\left[\left(j\frac{\omega}{\omega_v}\right)^2 + \left(\frac{\omega_{\infty 1}}{\omega_v}\right)^2\right]\left[\left(j\frac{\omega}{\omega_v}\right)^2 + \left(\frac{\omega_{\infty 2}}{\omega_v}\right)^2\right]\left[\ldots\right]} \right\} \tag{5}$$

with $|\Delta|_{min}$, which is the square root of the minimum magnitude of the bracketed polynomial, as yet undetermined.

Step 4. Take the sum of the squares of the real and imaginary parts of the bracketed polynomials to form the magnitude and by differentiation, or plotting versus the frequency variable, find the minimum numerical value of this magnitude. This minimum value is $|\Delta|^2_{min}$.

Step 5. The modified magnitude equation $|V_p/V|^2$ is now equal to the magnitude polynomial formed in Step *4* divided by the numerical value $|\Delta|^2_{min}$.

Step 6. Subtract from the modified magnitude equation of Step *5* the numerical value P_p/P_m. This quantity is the ratio of the power delivered to the load at the peak response frequency to the maximum power available from the resistive generator. For a resistive generator and a resistive load, an impedance-matched output is usually desired, so for this case P_p/P_m is usually set equal to 1.0; however P_p/P_m may be set equal to any numerical value from 1.0 down to zero. Absorb this numerical value in the modified magnitude equation of Step *5* to form a new numerator polynomial. (The denominator polynomial will remain unchanged.)

Step 7. Solve the numerator polynomial obtained in Step *6* for its $2n$ complex zeros. Multiply these ω/ω_v zeros by j to make them functions of $j(\omega/\omega_v)$.

Step 8. Use the n left-half-plane zeros to form a complex numerator polynomial in $j(\omega/\omega_v)$ having numerical coefficients. This complex polynomial is the numerator of (6).

$$\left\{\left|\frac{V_p}{V}\right|^2 - \left|\frac{P_p}{P_m}\right|\right\}\left(j\frac{\omega}{\omega_v}\right) = \frac{1}{|\Delta|_{min}} \left\{ \frac{\left(j\frac{\omega}{\omega_v}\right)^n + V_{n-1}\left(j\frac{\omega}{\omega_v}\right)^{n-1} + V_{n-2}\left(j\frac{\omega}{\omega_v}\right)^{n-2} + \ldots + V_0}{\left[\left(j\frac{\omega}{\omega_v}\right)^2 + \left(\frac{\omega_{\infty 1}}{\omega_v}\right)^2\right]\left[\left(j\frac{\omega}{\omega_v}\right)^2 + \left(\frac{\omega_{\infty 2}}{\omega_v}\right)^2\right]\left[\ldots\right]} \right\}. \tag{6}$$

Step 9. From the numerator polynomial formed in Step *3* and that formed in Step *8*, form the function

$$\left\{ \frac{2\left(j\frac{\omega}{\omega_v}\right)^n + (U_{n-2} + V_{n-2})\left(j\frac{\omega}{\omega_v}\right)^{n-2} + \ldots}{(U_{n-1} + V_{n-1})\left(j\frac{\omega}{\omega_v}\right)^{n-1} + (U_{n-3} + V_{n-3})\left(j\frac{\omega}{\omega_v}\right)^{n-3} + \ldots} \right\}. \tag{7}$$

Step 10. The function formed in Step *9* is any one of the following four input immittances of the required lossless network: when n is odd, it is $Z_{in\,sc}/R_a$ and $Y_{in\,oc}/G_a$: when n is even, it is $Y_{in\,sc}/G_a$ and $Z_{in\,oc}/R_a$; where R_a is the resistive termination on the left side of the network. Knowing the locations of the real frequency infinities as given in the denominator of (5), expand the function in continued-fraction form to obtain the network and element values in terms of R_a. When performing the expansion, it is necessary to know all the frequencies of infinite attenuation.

Step 11. When the right-half-plane zeros of Step *7* are used to form the polynomial of Step *8*, then the 2nd, 4th, 6th, et cetera, terms will be negative. If we use this polynomial to form the input immittance function, we obtain

$$\left\{\frac{2\left(j\frac{\omega}{\omega_v}\right)^n + (U_{n-2}+V_{n-2})\left(j\frac{\omega}{\omega_v}\right)^{n-2} + \ldots}{(U_{n-1}-V_{n-1})\left(j\frac{\omega}{\omega_v}\right)^{n-1} + (U_{n-3}-V_{n-3})\left(j\frac{\omega}{\omega_v}\right)^{n-3} + \ldots}\right\}. \tag{8}$$

Step 12. This function gives the expansion of the network in Step *10* from the other end, i.e. for n odd, this function is $Z_{\text{in sc}}/R_b$ and $Y_{\text{in oc}}/G_b$, respectively, and for n even, this function is $Z_{\text{in oc}}/R_b$ and $Y_{\text{in sc}}/G_b$, respectively. Assigning the infinite attenuation frequencies to the same network elements as in Step *10*, expand this function in continued-fraction form to obtain the same network as that in Step *10* but with element values given in terms of R_b, the termination on the right side of the network.

Step 13. Equating the two expressions for the same element, i.e. one in terms of R_a and the other in terms of R_b, will give the required ratio of R_b/R_a; all the reactive-element values and the terminations have now been synthesized. When the uniform loss used in Step *2* is now added to this network, it will produce the desired attenuation shape that was used in Step *1*.

Note: If the normalized decrement of Step *2* is very much smaller than the smallest real coordinate obtained in Step *1*, i.e. if the network elements to be used are essentially lossless as is the case in this paper, then Steps *2*, *4*, and *5* can be omitted, which materially reduces the amount of numerical work that must be done.

It should also be noted that when n is larger than 5 or 6, a large number of significant figures must be maintained in the simple partial-fraction expansion of Step *10* and Step *12*. For the cases of large n, the procedure outlined in Table 1 on page 298 of Dr. Darlington's paper[6] should be used.

6. *Need for Closed-Form Design Equations*

The procedures outlined in Sections 5.1 and 5.2 enable the engineer to synthesize a filter of any number of arms n if the numerical values contain sufficient significant figures; practically, this requires the use of a calculating machine and the time necessary to do the routine work.

The need for a large number of significant figures for all numerical values is particularly annoying and so it would be of real service to the practicing engineer if it were possible to obtain from the procedures of Sections 5.1 and 5.2 closed-form general solutions for the required element values.

These closed-form solutions would be obtained by using the general expressions for the roots in Step *3* of both procedures so that general expressions are obtained for the coefficients in the polynomials of (5) and (6), then, when the concluding steps are performed, it may be possible to recognize a law of formation for the required element-value equations.

7. *Element Values Required to Produce the Butterworth Attenuation Shape of (4)*

In January of 1931, E. L. Norton of the Bell Telephone Laboratories was granted a patent[7] showing that he had accomplished this general solution discussed in Section 6 for the attenuation shape of (4), when the low-pass network of Figure 5 has a resistance on one side only, the other side of the network being open-circuited if it ends in a capacitance or short-circuited if it ends in an inductance. He used the procedure of Section 5.1, and for the band-pass case his solution is as given in (9),

$$\frac{d_1}{\mathrm{BW}_{3db}/f_0} = \frac{1}{\sin(90°/n)},\quad d_{2\to n} = 0, \tag{9A}$$

$$\left(\frac{K_{r(r+1)}}{\mathrm{BW}_{3db}/f_0}\right)^2 = \frac{\cos^2[r(90°/n)]}{[\sin(2r-1)(90°/n)][\sin(2r+1)(90°/n)]}. \tag{9B}$$

Then in March of 1932, W. R. Bennett also of Bell Telephone Laboratories was granted a patent[8] showing that he had accomplished the general solution for the attenuation shape of (4), when the network has equal resistances on both sides. He also used the procedure of Section 5.1 and for the band-pass case his solution is as given in (10).

Using the procedure of Section 5.1 and keeping the expressions for the polynomial coefficients as general as possible, the writer was not able to recognize the law of formation for the coefficient-of-coupling values. However, using the procedure of Section 5.1 as described in detail in reference 1 and using numerical values for all the angle functions, it was very simple to obtain the relations, given in Tables 1 and 2, between the coefficients of coupling required for the Butterworth shape and those required for the Chebishev shape.

$$\left(\frac{d_{1,n}}{\mathrm{BW}_{3\mathrm{db}}/f_0}\right) = \frac{1}{2\sin(90°/n)}, \quad d_{2\rightarrow(n-1)} = 0, \tag{10A}$$

$$\left(\frac{K_{r(r+1)}}{\mathrm{BW}_{3\mathrm{db}}/f_0}\right)^2 = \frac{1}{4[\sin(2r-1)(90°/n)][\sin(2r+1)(90°/n)]}. \tag{10B}$$

8. Element Values Required to Produce Chebishev Attenuation Shape of (3)

In the two decades that have passed since Norton and Bennett achieved their closed-form solutions for the Butterworth response shape, no one accomplished—or at an rate no one published—the closed-form solution for the more-general Chebishev response shape of which the Butterworth shape is the limiting case.

In a series of letter discussions with Mr. V. D. Landon of Radio Corporation of America during the summer of 1952, this problem was considered and the general solution for the Chebishev attenuation shape was obtained by the following means.

[8] W. R. Bennett, United States Patent 1 849 656; March, 1932.

The relations in Table 1 are obtained by first solving the "Design Equations—Group 3" of reference 1 for the coupling values required for the Butterworth attenuation shape with the 3-decibel-down bandwidth as the reference bandwidth and with d_2 to d_n, inclusive, set equal to zero. Then the "Design Equations—Group 5" in reference 1 were solved for the coupling values required for the Chebishev attenuation shape—also with d_2 to d_n, inclusive, set equal to zero. Here, it should be noted the reference bandwidth is the "valley-decibel-down" bandwidth, not the 3-decibel-down bandwidth.

TABLE 1

COEFFICIENTS OF COUPLING FOR THE REACTIVE-LOAD CASE

$n = 2$	$\frac{(K_{12}/F_v)^2_C}{(K_{12}/F_{3db})^2_B} = S_2^2 + \cos^2(90°/2)$
$n = 3$	$\frac{(K_{12}/F_v)^2_C}{(K_{12}/F_{3db})^2_B} = S_3^2 + 0.2500$
	$\frac{(K_{23}/F_v)^2_C}{(K_{23}/F_{3db})^2_B} = S_3^2 + 1.000$
$n = 4$	$\frac{(K_{12}/F_v)^2_C}{(K_{12}/F_{3db})^2_B} = S_4^2 + 0.1464$
	$\frac{(K_{23}/F_v)^2_C}{(K_{23}/F_{3db})^2_B} = S_4^2 + 0.500$
	$\frac{(K_{34}/F_v)^2_C}{(K_{34}/F_{3db})^2_B} = S_4^2 + 0.8535$

TABLE 2

COEFFICIENTS OF COUPLING FOR THE RESISTIVE-LOAD RESISTIVE-GENERATOR CASE

$n = 2$	$\frac{(K_{12}/F_v)^2_C}{(K_{12}/F_{3db})^2_B} = S_2^2 + 1.000$
$n = 3$	$\frac{(K_{12}/F_v)^2_C}{(K_{12}/F_{3db})^2_B} = S_3^2 + 0.7500$
	$\frac{(K_{23}/F_v)^2_C}{(K_{23}/F_{3db})^2_B} = S_3^2 + 0.7500$
$n = 4$	$\frac{(K_{12}/F_v)^2_C}{(K_{12}/F_{3db})^2_B} = S_4^2 + 0.500$
	$\frac{(K_{23}/F_v)^2_C}{(K_{23}/F_{3db})^2_B} = S_4^2 + 1.000$
	$\frac{(K_{34}/F_v)^2_C}{(K_{34}/F_{3db})^2_B} = S_4^2 + 0.500$

The relations in Table 2 are obtained by first solving the "Design Equations—Group 3" of reference 1 for the coupling values required for the Butterworth attenuation shape with the 3-decibel-down bandwidth as the reference bandwidth and with d_2 to d_{n-1}, inclusive, set equal to zero. Next the "Design Equations—Group 5" for the Chebishev shape were solved with the "valley-decibel-down" bandwidth as the reference bandwidth, and d_2 to d_{n-1} set equal to zero.

With the above relations established and realizing that the numerical values in the relations are formed from combinations of the sin and cos of multiples of $(90°/n)$, it was a relatively simple matter to discover that function of $(90°/n)$ that produced the numerical values in the two tables.

For the reactive-load case, the relation for the double-tuned circuit, i.e. $n=2$, could actually be made in closed form as given in Table 1, and using this as the starting point it required a relatively small amount of work to see that all the ratios in Table 1 were given by the simple function of (11).

$$\frac{(K_{r(r+1)}/F_v)_C^2}{(K_{r(r+1)}/F_{3db})_B^2} = S_n^2 + \sin^2\left(r\frac{90°}{n}\right). \quad (11)$$

Actually, due to the form of the double-tuned-circuit solution, the angle function first obtained was $\cos^2 (n - 2r)\ (90°/n)$; it was then realized that this was identical to the function given in (12).

Since Norton and Bennett had obtained the solutions for $(K_{r(r+1)}/F_{3db})_B$ as given in Section 7, the relations of (11) and (12) now give us immediately the desired solutions for the coefficients of coupling required for the general Chebishev attenuation shape.

Actually, due to the hint given by the double-tuned-circuit relation, the angle function first obtained was $\cos^2 (n - r)\ (90°/n)$; it was then realized that this was identical to the function given in (11).

For the resistive-load resistive-generator case, it was immediately noted that the numbers in Table 2 were the squares of 1.0, 0.866, and 0.707, which in conjunction with the hint given by the form of the solution for the double-tuned circuit led to the realization that all the ratios in Table 2 were given by the simple function (12).

$$\frac{(K_{r(r+1)}/F_v)_C^2}{(K_{r(r+1)}/F_{3db})_B^2} = S_n^2 + \sin^2\left(2r\frac{90°}{n}\right). \quad (12)$$

TO OBTAIN THE SHAPE $\left(\frac{V_p}{V}\right)^2 = 1 + \left[\left(\frac{V_p}{V_v}\right)^2 - 1\right] \cosh^2\left\{n \cosh^{-1}\frac{\text{BW}}{\text{BW}_v}\right\}$

Resistive Generator and Resistive Load

$$\frac{Q_{1,n}}{f_0/\text{BW}_v} = \frac{2\sin\theta}{S_n} \qquad Q_{2\to(n-1)} = \infty$$

$$\left[\frac{K_{r(r+1)}}{\text{BW}_v/f_0}\right]^2 = \frac{[S_n^2 + \sin^2 2r\theta]}{4\{\sin(2r-1)\theta\}\{\sin(2r+1)\theta\}}$$

Resistive Generator and Reactive Load (or vice versa)

$$\frac{Q_1}{f_0/\text{BW}_v} = \frac{\sin\theta}{S_n} \qquad Q_{2\to n} = \infty$$

$$\left[\frac{K_{r(r+1)}}{\text{BW}_v/f_0}\right]^2 = \frac{[S_n^2 + \sin^2 r\theta]}{\sec^2(r\theta)\{\sin(2r-1)\theta\}\{\sin(2r+1)\theta\}}$$

$$\theta = \frac{90°}{n} \qquad S_n = \sinh\left\{\frac{1}{n}\sinh^{-1}[(V_p/V_v)^2 - 1]^{-1/2}\right\}$$

Figure 7—The new design equations for nondissipative inverse-arm filters.

For the required input and output decrement values, the same procedure used to obtain the values given in Tables 1 and 2 showed immediately that for both the reactive-load case and the resistive-load resistive-generator case the relation between the decrement required for the Chebishev shape and that required for the Butterworth shape is as given in (13).

$$\frac{(d/F_v)_C}{(d/F_{3db})_B} = S_n. \tag{13}$$

Equations (11), (12), and (13) combined with the solutions of Section 7 are the desired design equations, and they have been combined to form Figure 7; they are the two new sets of equations for the design of the constant-k-configuration filters that are referred to in the title of this paper.

As indicated at the top of the figure, the attenuation shape that will be obtained will be the optimum Chebishev attenuation shown in Figure 6C. It is important to realize that the design equations of Figure 7 are given in terms of the "valley bandwidth" (BW_v) which is the bandwidth between the points on the skirt that are down by the same number of decibels as the peak-to-valley ratio V_p/V_v. The quantity S_n is a function of the number of resonators n used and the peak-to-valley ratio desired, and the reader should note that as V_p/V_v approaches unity, S_n becomes very large and therefore the required K's will be a large number of times the fractional *valley* bandwidth. However, a required bandwidth at some other decibels down rather than at the valley-decibels down is very often specified, and it is therefore necessary to get the numerical relation between the valley-decibels-down bandwidth and the specified-decibels-down bandwidth by using the shape equation at the top of Figure 7.

9. *Application of Design Equations to the Low-Pass Ladder*

As previously indicated, if one writes the transfer-impedance equations for the low-pass ladder, compares them to those obtained for the band-pass case, and uses a suitable normalizing procedure, it is found that in the low-pass case the frequency variable ω (i.e., $2\pi f$) is exactly equivalent to the band-pass-case frequency variable

$$\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \equiv \frac{BW}{f_0},$$

and in the low-pass case the quantity

$$1/(L_{series}C_{shunt})^{1/2}$$

is exactly equivalent to the well-known coefficient of coupling of band-pass-coupled-circuit theory; and the quantity L/R in a series arm and RC for a shunt arm are exactly equivalent to the well-known resonant-frequency Q of band-pass circuit theory. Thus to apply the equations of Figure 7 to a low-pass ladder, we do the following:

In place of BW_v/f_0 use ω_v.

In place of Q_1 use L_1/R_1 or R_1C_1.

In place of $K_{r(r+1)}$ use

$$1/(L_rC_{r+1})^{1/2}$$

or

$$1/(C_rL_{r+1})^{1/2}.$$

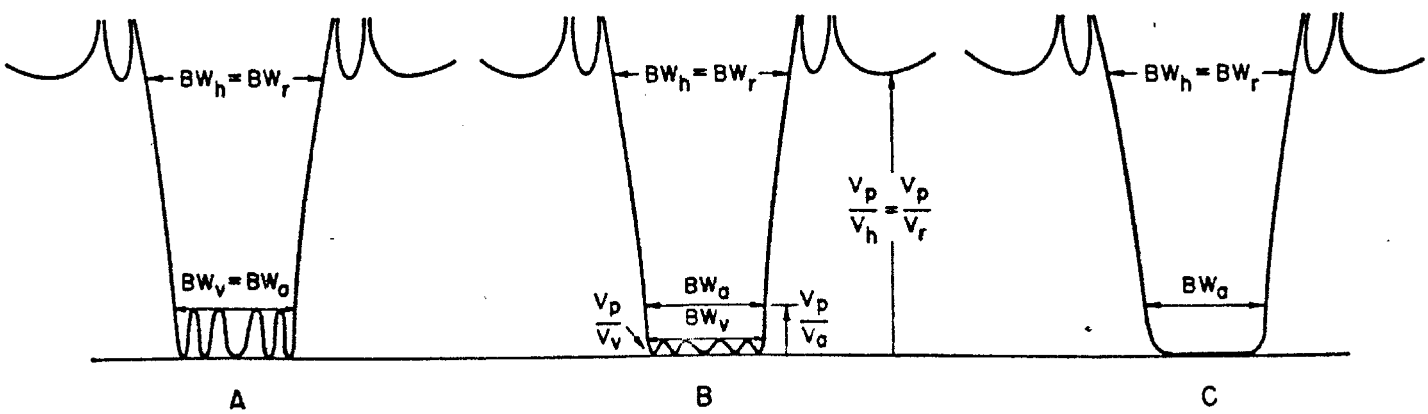

Figure 8—The optimum elliptic-function response shapes for m-derived-configuration filters. See Section 10.

10. *Corresponding Problem for m-Derived Configurations*

Exactly equivalent to the optimum attenuation shape of (3) for constant-k-configuration filters, there is an optimum attenuation shape for m-derived-configuration filters. This shape is shown in Figure 8 and is optimum in the sense that for a given allowable ripple in the accept band, a given minimum attenuation in the reject band (for a width equal to 10 accept bandwidths), and a given number of arms, it produces the sharpest possible rate of cutoff between the accept and reject bands.

Still to be discovered are the general design equations equivalent to those of Figure 7, which give the element values required to produce the optimum shapes to be described below. Just as the Chebishev shape of (3) changes in form for the limiting case of 0-decibel ripples and becomes (4), so does the optimum m-derived-shape equation change in form. Section 10.1 will consider the general equation and Section 10.2 will consider the limiting case of 0-decibel ripples in the pass band.

10.1 Optimum Attenuation Shape for m-Derived-Configuration Filters

All the following discussion will relate to the band-pass case and it will be assumed that the reader realizes it is immediately applicable to the low-pass case when the frequency variable ω is used in place of BW/f_0. As evolved by Norton and Darlington, plus a simple modification, the optimum shape is given by (14).

$$\left|\frac{V_p}{V}\right|^2 = 1 + \left(\left|\frac{V_p}{V_v}\right|^2 - 1\right)\left(\frac{cn}{dn}\right)_v \left[n\frac{K_v}{K_f}\left(\frac{cn}{dn}\right)_f^{-1}\frac{BW}{BW_v}\right], \quad (14A)$$

where

$$v = \left[\frac{(V_p/V_v)^2 - 1}{(V_p/V_h)^2 - 1}\right]^{1/2} \quad (15)$$

$$f = \frac{BW_v}{BW_h}. \quad (16)$$

For a given number of arms n, in (14), v and f cannot be picked independently but must always satisfy the relation between the three quantities given by (17).

$$\log q_v = n \log q_f, \quad (17)$$

where q is the so-called modular constant of the modulus given by the subscript. The function $\log q_k$ is tabulated on pages 49–51 of the 1945 edition of Jahnke and Emde; and the Smithsonian Elliptic Function Tables compiled by S. W. and R. M. Spenceley contains a very useful short appendix dealing with the numerical computation of the various elliptic functions. (It will be noted that for modulus values k less than 0.1 (say), the corresponding modular constant is simply $q_k = (k/4)^2$.) In (14), the symbol cn/dn stands for the ratio of the two elliptic functions cn over dn, where the subscript v or f is the modulus of the function; and K_v and K_f are the complete elliptic integrals of the first kind, evaluated for the modulus value given by the subscript.

It is evident from Figure 8 that the modulus v in (15) is immediately set by *two voltage ratios*, i.e. the maximum allowable ripple in the pass band V_p/V_v and the minimum required attenuation in the reject band V_p/V_h. Similarly, the modulus f in (16) is immediately set by the ratio of *two bandwidths*, i.e. the valley bandwidth BW_v and the "hill" bandwidth BW_h, which are the two bandwidths on the attenuation skirts where the attenuation is equal to the peak-to-valley ratio V_p/V_v and peak-to-hill-ratio V_p/V_h, respectively.

Equation (14A) takes on two different forms depending on the part of the attenuation characteristic in which one is interested. From the middle of the pass band out to the valley bandwidth, i.e. when BW/BW_v is less than unity, (14A) stays as given. In the cutoff region between the edge of the valley bandwidth and the edge of the hill bandwidth, i.e. where BW/BW_v varies between unity and $1/f$, (14A) turns into (14B).

$$\left|\frac{V_p}{V}\right|^2 = 1 + \frac{(V_p/V_v)^2 - 1}{dn_{v'}[n(K_v/K_f)dn_{f'}^{-1}(BW_v/BW)]}. \quad (14B)$$

The prime indicates the complimentary modulus, i.e. $v' = (1 - v^2)^{1/2}$ and $f' = (1 - f^2)^{1/2}$. (Note also the inversion of the bandwidth ratio in the bracketed expression.)

Finally outside the hill bandwidth, i.e. where $\mathrm{BW}/\mathrm{BW}_v$ is greater than $1/f$, (14A) turns into (14C).

$$\left|\frac{V_p}{V}\right|^2 = 1 + \frac{(V_p/V_h)^2 - 1}{(cn/dn)_{v'}{}^2[n(K_v/K_f)(cn/dn)_{f'}{}^{-1}(\mathrm{BW}_h/\mathrm{BW})]}, \quad (14C)$$

and of course the relations of (15), (16), and (17) apply to (14A), (14B), and (14C).

In the above equations, the writer has used the (cn/dn) elliptic function rather than the (sn) elliptic function because with the former a single equation (14) can be written that holds for both n odd *and* n *even*. Similarly, the important zero- and pole-location equations that follow will apply to both n odd and n even. Insofar as numerical work is concerned, the fact that $(cn/dn)u = sn\,(K - u)$ enables the cn/dn values to be obtained from the more-common sn tables.

It should be mentioned that when an even number of arms n is to be used, (14) calls for a finite attenuation at infinite frequency. To produce this phenomenon, it is necessary to make the numeric P_p/P_m in Step *6* of the Darlington procedure equal to zero; this will then produce a short-circuited termination for a network ending in a series arm, or an open-circuited termination for a network ending in a shunt arm, which will in turn produce finite attenuation at infinite frequency.

For both the direct procedure and the Darlington procedure, it is necessary to solve (14) for its $j(\mathrm{BW}/\mathrm{BW}_v)$ zeros; the left-half-plane zeros are given in (18), (18A), and (18B).

$$j(\mathrm{BW}_0/\mathrm{BW}_v)_m = -r_m{}^c \pm j i_m{}^c \quad (18A)$$

$$r_m{}^c = \frac{[(sn/cn)_{f'}B^c][f'^2(sn/dn^2)_f A_m{}^c]}{1+f^2[(sn/cn)_{f'}{}^2B^c][(cn/dn)_f{}^2A_m{}^c]} \quad (18B)$$

$$i_m{}^c = \frac{[(dn/cn^2)_{f'}B^c][(cn/dn)_f A_m{}^c]}{1+f^2[(sn/cn)_{f'}B^c][(cn/dn)_f{}^2A_m{}^c]}, \quad (18C)$$

where the angles $A_m{}^c$ and B^c are given by (19).

$$A_m{}^c = (2m-1)(K_f/n) \quad (19A)$$

$$B^c = \frac{1}{n}\frac{K_f}{K_v}\left(\frac{sn}{cn}\right)_{v'}^{-1}\left[\left(\frac{V_p}{V_v}\right)^2 - 1\right]^{-1/2}. \quad (19B)$$

It should be realized that, for modulus values as close to unity as is v' in practical cases, one can use the approximation $(sn/cn)^{-1} = \sinh^{-1}$.

Thus for lossless elements, the numerator polynomial of (5) required by Step *3* of both the direct and Darlington procedures is obtained by multiplying together a total of n factors each of the form

$$[j(\mathrm{BW}/\mathrm{BW}_v) - (-r_m{}^c \pm j i_m{}^c)],$$

where $r_m{}^c$ and $i_m{}^c$ are given by (18) and (19). This procedure will now give us the U coefficients of (5).

Next needed for the Darlington procedure are the zeros of the equation formed when the numeric P_p/P_m is subtracted from (14); the two most-common cases are considered in the next two paragraphs.

For resistive loading in both sides, impedance-matched output is usually desired at the peaks, and for this case $P_p/P_m = 1.0$. The $j(\mathrm{BW}/\mathrm{BW}_v)$ zeros required in Step *7* of the Darlington procedure are then for this impedance-matched case given by (20).

$$\left(j\frac{\mathrm{BW}_0}{\mathrm{BW}_v}\right)_m = 0 \pm j\left(\frac{cn}{dn}\right)_f(2m-1)\frac{K_f}{n}. \quad (20)$$

Thus the numerator polynomial of (6) required by Step *8* of the procedure is obtained by multiplying together a total of n factors each of the form

$$[j(\mathrm{BW}/\mathrm{BW}_v) - (\pm j i)],$$

where i is value of the j term in (20). This procedure will now give us the V coefficients of (6).

The input-immittance equations (7) and (8) can now be formed.

For a reactive load on one side of the network, $P_p/P_m = 0$, and so the zeros of Step *7* will be identical to those used for Step *3*, and thus the V

coefficients of (6) will be identical to the U coefficients of (5). Thus for this case as soon as the U's are obtained, (7) can be formed.

In the continued-fraction expansion procedure, it is necessary to know the frequencies of infinite attenuation and to assign these frequencies to specific arms of the network; the infinite-attenuation frequencies of (14) are given by (21).

$$\left(j\frac{BW_\infty}{BW_v}\right)_m = 0 \pm j\frac{1}{f(cn/dn)_j A_m^c}. \quad (21)$$

It has not yet been possible to recognize the law of formation that would give a closed-form solution for the network values.

10.2 Limiting Case of No Ripples in the Pass Band of m-Derived-Configuration Filters

When the peak-to-valley ratio V_p/V_v in (14B) is made to approach 0 decibels and the equation is expressed in terms of the hill bandwidth BW_h instead of valley bandwidth, then in the limit the response equation becomes (22) and the shape is that of Figure 8C.

$$\left|\frac{V_p}{V}\right|^2 = 1 + \frac{(V_p/V_h)^2 - 1}{\cosh^2[n\cosh^{-1}(BW_h/BW)]}. \quad (22)$$

It is of interest to note that if one applies (22) to two points on the skirt and allows the V_p/V_h ratio to become infinite, then the Butterworth equation (4) results. Similarly, (14) would turn into the Chebishev equation (3) if the V_p/V_h ratio is correctly made to approach infinity.

Paralleling the paragraphs of Section 10.1 to obtain the required element values, we must solve (22) for its $j(BW/BW_h)$ zeros; the n left-half-plane zeros are given by (23), (23A), and (23B).

$$j(BW_0/BW_h) = -r_m^b \pm j i_m^b \quad (23)$$

where

$$r_m^b = \frac{\sinh B^b \sin A_m^b}{\sinh^2 B^b + \cos^2 A_m^b} \quad (23A)$$

and

$$i_m^b = \frac{\cosh B^b \cos A_m^b}{\sinh^2 B^b + \cos^2 A_m^b} \quad (23B)$$

where the angles A_m^b and B^b are given by (24).

$$\left.\begin{aligned} A_m^b &= (2m-1)(90°/n) \\ B^b &= (1/n)\sinh^{-1}[(V_p/V_h)^2 - 1]^{1/2}. \end{aligned}\right\} \quad (24)$$

Thus the numerator polynomial of (5) required by Step *3* of both the direct and Darlington procedures is obtained for lossless elements by multiplying together a total of n factors of the form

$$[j(BW/BW_h) - (-r_m^b \pm j i_m^b)],$$

where r_m^b and i_m^b are given by (23). This procedure will now give us the required U coefficients of (5).

Next needed are the zeros of the equation formed when the numeric P_p/P_m is subtracted from (22).

For the impedance-matched case where $P_p/P_m = 1.0$, the $j(BW/BW_h)$ zeros of Step *7* are all zero, so that the numerator polynomial (6) of Step *8* is simply the one term $j(BW/BW_h)^n$ and all the V coefficients are zero. Thus for this attenuation shape and for the impedance-matched case, the input-immittance equations (7) and (8) can be formed as soon as the U's are obtained.

For the reactive-load case, $P_p/P_m = 0$ and so the V coefficients of (6) are identical to the U coefficients of (5), and as soon as the U's are obtained the input immittance (7) can be written.

As in Section 10.1, it is necessary to know the frequencies of infinite attenuation and to assign these frequencies to specific arms of the network. These infinite-attenuation frequencies of (22) are given by (25).

$$\left(j\frac{BW_\infty}{BW_h}\right)_m = 0 \pm j\frac{1}{\cos A_m^b}. \quad (25)$$

It has not yet been possible to recognize the law of formation that would give a closed-form solution for the network values.

11. Postscript—Recent Publications on this Subject

In Section 8, it was mentioned that in the 20 years that have passed since Norton and Bennett obtained their solutions, no one published the more-complicated Chebishev solutions. The attention of the writer has been directed to the following three papers whose existence emphasizes the often-repeated phenomenon that when the time is ripe, a problem is usually solved almost simultaneously in many parts of the world.

Ernest Green has apparently accomplished the most, for in his paper,[9] equations (2), (3), (5), and (6) give the general solutions for the Butterworth and Chebishev responses for *any ratio of terminations* instead of merely for equal loading on both sides or for loading on one side only.

The first paper to be published with the general solution for the case of equal loading on both sides is that of Vitold Belevitch.[10]

Finally, H. J. Orchard[11] using an "intuitive" process that was probably similar to that used in Section 8 of this paper, obtained the solution for the case of loading on one side only.

12. *Acknowledgment*

The author takes this opportunity to express his appreciation to Mr. Georges Deschamps of Federal Telecommunication Laboratories for many productive discussions with him on some of the material in this paper.

[9] E. Green, "Exact Amplitude Frequency Characteristics of Ladder Networks," *Marconi Review*, volume 16, number 108, pages 25–68; 1953.

[10] V. Belevitch, "Tchebyshev Filters and Amplifier Networks," *Wireless Engineer*, volume 29, pages 106–110; April, 1952.

[11] H. J. Orchard, "Formulae for Ladder Filters," *Wireless Engineer*, volume 30, pages 3–5; January, 1953.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \coth(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\operatorname{arsinh}(x) = \ln(x + \sqrt{x^2+1}) \qquad \operatorname{arcosh}(x) = \ln(x + \sqrt{x^2-1})$$

$$\operatorname{artanh}(x) = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \operatorname{arcoth}(x) = \tfrac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$