A note on transformer models

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In this note we derive some equtions for the description of transformers. We further show, that it makes no sense to speak of a primary- or secondary stray-inductance, since equivalent models can be derived that have no primary (resp. no secondary) leakage.

B.1 Transformer T-model

We view the transformer as a purely inductive two-port (no loss mechanisms nor capacitances are included). The primary port has the connectors 1 and 2, and the secondary port has connectors 3 and 4. The electrically equivalent T-model is based on a magnetizing inductance L_h, two stray inductances L_{1s} and L_{2s} and an ideal transformer with transformation ratio m. (Warning: This ratio m is not necessaraly the turns ratio as we will see !)

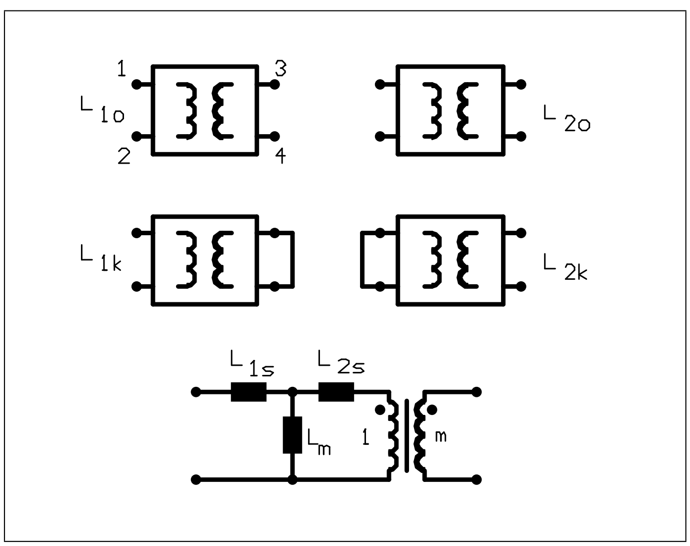

Fig. B.1 Transformer

These four internal components can not be measured directly, and we will see, that for a given transformer the equivalent model is not unique.

B.2 Measurements and equivalent circuits

Usually the simplest measurements that are performed in order to characterize the transformer are inductance measurements at one side while leaving the other side open or shortening the other side. So we get four measured quantities:

L_{1o} : Inductance of primary side, secondary side open

L_{1k} : Inductance of primary side, secondary side shortened

L_{2o} : Inductance of secondary side, primary side open

L_{2k} : Inductance of secondary side, primary side shortened

It is a common misconception, that using these four measured parameters it would be possible to uniquely determine the four quantities of the electrically equivalent model. This is not the case.

We first show, that the 4 measurements are not independent. Only three of the measurements are independent. We further show, that one of the four elements of the electrical model can be chosen arbitrarily to match the three independent parameters of a transformer. We use the transformation ratio m and show how the primary leakage L_{1s} and the secondary leakage L_{2s} depend on the particular value of m.

To see, why there are only three independent measurements we first write down the equations of the measured parameters depending on the parameters of the electrical model:

$$L_{1o} = L_{1s} + L_m \quad (I)$$

$$L_{1k} = L_{1s} + (L_m || L_{2s}) \quad (II)$$

$$L_{2o} = m^2(L_{2s} + L_m) \quad (III)$$

$$L_{2k} = m^2\{L_{2s} + (L_m || L_{1s})\} \quad (IV)$$

Here || denote the value of the parallel circuit .
By multiplying eqns. (I) and (III) resp. $II)$ and (IV) it is now easy to verify, that:

$$L_{2k}L_{1o} = L_{1k}L_{2o}$$

This shows the fundamental relation between the four measured parameters. If three of them are given, you can compute the fourth. So the electrical T-model is based on only three parameters, but consists of four components.

The next task is to express the values of the components of the electrical model as functions of the measured values. Since only three parameters are determined, we take m as a parameter. By subtracting (I) and (II) we get

$$L_{10} - L_{1k} = L_m - \frac{L_m L_{2s}}{L_m + L_{2s}} = \frac{L_m^2}{L_m + L_{2s}}$$

Now using equation (III) we get

$$L_{1o} - L_{1k} = \frac{L_m^2}{L_{2o}/m^2}$$

Therefore we can express the magnetizing inductance as function of the three measured values and the parameter m as follows:

$$L_m^2 = \frac{L_{2o}}{m^2}(L_{1o} - L_{1k})$$

Using equation (I) and (III) we can express the other two values of the electrical model:

$$mL_{1s} = mL_{1o} - \sqrt{L_{2o}(L_{1o} - L_{1k})}$$

$$mL_{2s} = \frac{L_{2o}}{m} - \sqrt{L_{2o}(L_{1o} - L_{1k})}$$

These two equations give a good insight, how the stray-inductance values of the electrical model depend on the parameter m: If m is too small, the primary stray inductance L_{1s} in the model becomes negative. If m is too big, the secondary stray inductance L_{2s} becomes negative. If m is chosen between these two values we get positive stray inductances. (It should be noted, that also the models containing negative stray inductances fulfill all equations and yield valid transformer models, one is even free to choose any complex number as m, the only problem is, that these model have no direct physical realisations as T-circuits). We give now formulas for some special cases:

Case 1: No primary stray inductance

If one chooses

$$m^2 = \frac{L_{2o} - L_{2k}}{L_{1o}}$$

we get $L_{1s} = 0$ and

$$L_{2s} = \frac{L_{1k}L_{2o}}{L_{2o} - L_{2k}}$$

and

$$L_m = L_{1o}$$

Case 2: No secondary stray inductance

If one chooses

$$m^2 = \frac{L_{2o}}{L_{1o} - L_{1k}}$$

we get $L_{2s} = 0$ and

$$L_{1s} = L_{1k}$$

and

$$L_m = L_{1o} - L_{1k}$$

Case 3: Primary equals secondary stray inductance

If one chooses

$$m^2 = \frac{L_{2o}}{L_{1o}}$$

we get

$$L_s = L_{1s} = L_{2s} = L_{1o}(1 - \sqrt{1 - L_{1k}/L_{1o}})$$

and

$$L_h = L_m = L_{1o}\sqrt{1 - L_{1k}/L_{1o}}$$

This case is useful when the transformer construction is symmetrical, and one also wants a symmetrical model to reflect that fact.

Usually the coupling-factor k is defined by using the symmetrical T-model. So we can define the coupling factor for the general transformer by:

$$k = \frac{L_h}{L_s + L_h}$$

Using the formulas from above we arive at

$$k = \sqrt{1 - L_{1k}/L_{1o}}$$

So the coupling factor determination needs only measurements from one side. Due to the fundamental relation

$$L_{1k}L_{2o} = L_{2k}L_{1o}$$

we get the result, that this coupling factor measured from the secondary side is the same:

$$k = \sqrt{1 - L_{2k}/L_{2o}}$$

This shows, that the coupling factor is really a characteristic of the transformer, not of the chosen model. Changing m or introducing a second ideal transformer does not change the coupling factor.

In our consderation we placed the ideal transformer on the secondary side. Clearly it can also be placed on the primary side. The formulas can simply be derived by interchanging primary and secondary indicies. The following figure shows a collection of electrically equivalent circuits for the values:

$$L_{10} = 100 \quad L_{1k} = 36 \quad L_{2o} = 1600 \quad L_{2k} = 576$$

(arbitrary units used).

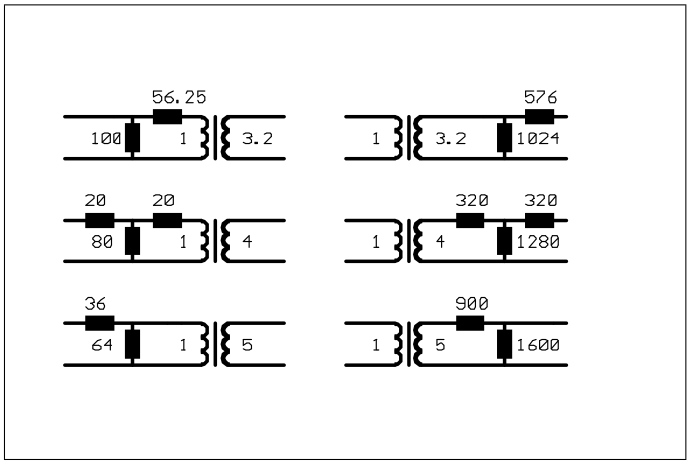

Fig. B.2 Equivalent transformers

B.3 Interpretation

It has been shown, that there are equivalent electrical T-models that are parametrized by m. For every transformer m can be chosen in a way that the primary or secondary leakage disappear in the T-model. Therefore it makes no sense to say, that a specific capacitor resonates with the primary or secondary stray inductance or with the magnetizing inductance of the transformer. This makes only sense with respect to the chosen T-model. By chosing the parameter m one can get equivalent descriptions, and sometimes its easier to understand a circuit when one of the leakage inductances is zero in the model. Therefore the equations derived are useful if one wants to analyse circuits because if one of the inductances is zero, normally its easier to write down the equations governing the circuits.

We have seen, that there existes a certain range for the parameter m within which we get positive stray inductances. If we denote m_1 the value obtained for case 1: no primary stray inductance, and similar m_2 is the value for case 2: no secondary stray inductannce, we can derive:

$$\frac{m_2}{m_1} = \frac{1}{k^2} \geq 1$$

where k again denotes the coupling coefficient. This shows, that in the case of perfect coupling $k = 1$ we are no longer free to choose m, in this case m is fixed and it is obvious, that in this case it is the turns-ratio, since perfect coupling can only be realized if both windings are identical in geometry and differ only in turn numbers. If the coupling is not perfect, we can determine only a range of possible turns-ratio of the ideal transformer of the T-model.

It has been shown, that in general the number m must not be the turns-ratio n of the transformer. It is even not clear, whether the turns-ratio n is within the range $m_1 \ldots m_2$ of parameter values leading to positive stray inductances. In all practical cases it has been observed, that $m_1 \leq n \leq m_2$.

B.4 Inductance coefficients

Models based on the magnetic fields are often based on the inductance coefficient matrix $L = (L_{ij})$. The transformer equations using these coefficients are (in the case of two windings)

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = j\omega \begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Using these inductance coeffiients one finds:

$$L_{1o} = L_{11} \qquad L_{1k} = \frac{L_{11}L_{22} - L_{12}^2}{L_{22}}$$

$$L_{2o} = L_{22} \qquad L_{2k} = \frac{L_{11}L_{22} - L_{12}^2}{L_{11}}$$

Further we find

$$m_1 = \frac{L_{12}}{L_{11}} \quad m_2 = \frac{L_{22}}{L_{12}}$$

The matrix of the inductance coefficients is positive semi-definite, this assures $L_{11} \geq 0$,$L_{22} \geq 0$ and $det(L) = L_{11}L_{22} - L_{12}^2 \geq 0$, so that this ensures positive values for L_{1k} and L_{2k}.

We further have (m given for symmetrical T-model):

$$k = \frac{L_{12}}{\sqrt{L_{11}L_{22}}} \qquad m = \sqrt{\frac{L_{22}}{L_{11}}}$$

B.5 Voltage transfer ratios

The voltage transfer ratio is also easily measurable. The relation to the other quantities can easily be understood by considering the symmetrical model (given by L_h,L_s,m) with inductances on the primary side.

We denote the primary voltage with U_1, the secondary voltage with U_2. In the case of unloaded secondary ($I_2 = 0$) we get for the voltage transfer ratio u_{12}:

$$u_{12} = \frac{U_2}{U_1}|_{I_2=0} = m\frac{L_h}{L_h + L_s} = m\ k$$

For unloaded primary we get

$$u_{21} = \frac{U_1}{U_2}|_{I_1=0} = \frac{1}{m}\frac{L_h}{L_h + L_s} = \frac{k}{m}$$

Together this means:

$$k = \sqrt{u_{12}u_{21}} \quad m = \sqrt{\frac{u_{12}}{u_{21}}}$$

This is a way to measure directly k and m without any inductance measurements !

B.6 Adjust measurement values

Using an inductance meter it is easy to measure the inductance coefficients

$$L_{1o}, L_{1k}, L_{2o}, L_{2k}$$

Due to measurement errors these willl not be the true values. We denote the measured values by

$$\hat{L}_{1o}, \hat{L}_{1k}, \hat{L}_{2o}, \hat{L}_{2k}$$

For the measured Values the error-quantity e will be not zero.

$$e = \hat{L}_{2k}\hat{L}_{1o} - \hat{L}_{1k}\hat{L}_{2o}$$

We now want to adjust the measured values so that e is zero for the new values. We use a correction factor $\alpha \approx 1$

$$L_{1o} = \hat{L}_{1o}\alpha \quad L_{1k} = \hat{L}_{1k}/\alpha \quad L_{2o} = \hat{L}_{2o}/\alpha \quad L_{2k} = \hat{L}_{2k}\alpha$$

$$e = \alpha^2 \hat{L}_{2k}\hat{L}_{1o} - \hat{L}_{1k}\hat{L}_{2o}/\alpha^2$$

To mkake e zero we put

$$\alpha = \left(\frac{\hat{L}_{1k}\hat{L}_{2o}}{\hat{L}_{2k}\hat{L}_{1o}}\right)^{1/4}$$

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Example computation by LUA.....
measured:
L1o=  122.000 uH
L1k=   22.000 uH
L2o=  120.000 uH
L2k=   22.000 uH
error=L2k*L1o-L1k*L2o=      44.000000 uH^2
alphaCorrection=     0.995876
Values adjusted...
L1o=  121.497 uH
L1k=   22.091 uH
L2o=  120.497 uH
L2k=   21.909 uH
L2k*L1o-L1k*L2o=      -0.000000 uH^2
mNoPrim=     0.900800
mNoSek =     1.100987
mSymm  =     0.995876
k      =     0.904531
m=mSymm   =  0.995876
Lm   =  109.898 uH  L1s =   11.599 uH  L2s =   11.599 uH
m=mNoPrim =  0.900800
Lm   =  121.497 uH  L1s =    0.000 uH  L2s =   27.000 uH
m=mNoSek  =  1.100987
Lm   =   99.406 uH  L1s =   22.091 uH  L2s =    0.000 uH
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