The calculation of coupling gap and fringing capacitances in coupled rectangular bars between ground planes

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Abstract

We consider a structure comprising an array of coupled rectangular bars between ground planes, in which it is frequently desired to know the appropriate spacing to realise a given bar-to-bar coupling capacitance per unit length. The converse problem of finding the coupling capacitance given the spacing has been treated by Getsinger in [2]*, in which the results were presented in graphical form, not easy to incorporate into a computer program. This paper presents a new analysis which solves the problem directly, in a form capable of being implemented numerically. The algorithms can also be extended to calculate the fringing capacitances.*

1. Introduction

Coupled strips between ground planes of the type shown in figure 1 have been used for many years as essential elements of many r.f. devices such as filters (*e.g.* combline and interdigital) and directional couplers. The theoretical treatment of such subjects is derived from basic transmission line theory.

A simple transmission line structure comprising a single conductor surrounded by an outer conducting enclosure will support a transverse electromagnetic (TEM) wave. Its idealised performance can be discussed entirely in terms of its inductance, L, and capacitance, C, per unit length: throughout this paper the term 'capacitance' is used to mean the capacitance per unit length of strip. These fundamental parameters can be alternatively expressed in terms of other parameters, namely, characteristic impedance, Z_0 (or characteristic admittance Y_0),

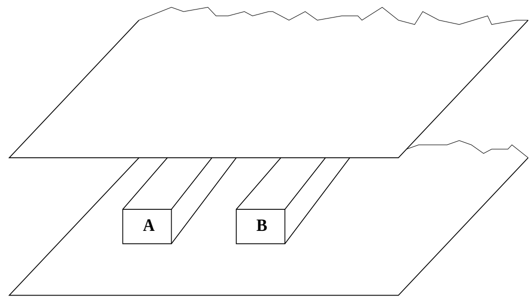

Figure 1: Parallel bars between ground planes forming a pair of coupled transmission lines.

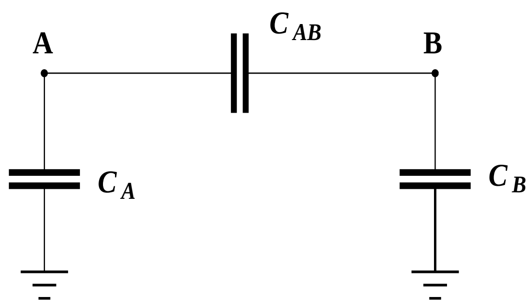

Figure 2: Equivalent circuit of a pair of coupled transmission lines A and B, showing capacitances per unit length.

velocity of wave propagation v and the relative dielectric constant ϵ_r of the medium surrounding the conductors (assumed uniform). The relationships between these parameters are given by

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L}{C}} = vL = \frac{1}{vC} \tag{1}$$

Additionally, $v = 1/\sqrt{LC} = c/\epsilon_r$ where c is the velocity of light in free space. Thus any two of the set of parameters $\{Z_0$ or Y_0, v or ϵ_r, L, $C\}$ are sufficient to specify system performance. An equivalent circuit of a pair of coupled transmission lines A and B in terms of the capacitances can be reduced to that of figure 2, in which the definitions of characteristic impedance and admittance are extended as follows. An even mode characteristic admittance Y_{0e}^A of a pair of transmission lines is defined as that of a given line, A, when the other line, B, is driven by an equal in-phase voltage. An odd mode characteristic admittance Y_{0o}^A is that of line A, when line B is driven by an equal, phase-reversed voltage.

$$Y_{0e}^A = vC_A \qquad \text{and} \qquad Y_{0o}^A = v(C_A + 2C_{AB}) \tag{2}$$

The 'primary mode' characteristic admittance Y_0 is defined with all other elements earthed. Hence in terms of the quantities C_A and C_{AB} [1]

[1] It should be noted in passing that these definitions do not translate directly into definitions of characteristic

$$Y_0^A = v(C_A + C_{AB}) \tag{3}$$

Many commonly used configurations [3] of two coupled lines require that the values of self (C_A, C_B) and mutual (C_{AB}) capacitance be precisely determined. The problem was first essayed in passing in [1]. The mathematical treatment of the problem is given in the appendix of that paper. In the 1950s and –60s the requirement for a solution to the capacitance problem became urgent and the standard treatment was published in [2] for rectangular lines. The results were presented in graphical form and some skill is required in interpolation. Additionally, graphical results do not lend themselves to incorporation in computer programs. It was decided therefore to investigate the possibility of producing algorithms to give required results in a filter design computer program.

It has been found possible to determine the dimensions and spacing of bars to produce the required values of self- and mutual-capacitance in a form suitable for computer evaluation. This matter is dealt with in detail below. The nomenclature adopted has been chosen to be the same as that of [2] to facilitate comparisons.

2. Coupling between rectangular bars

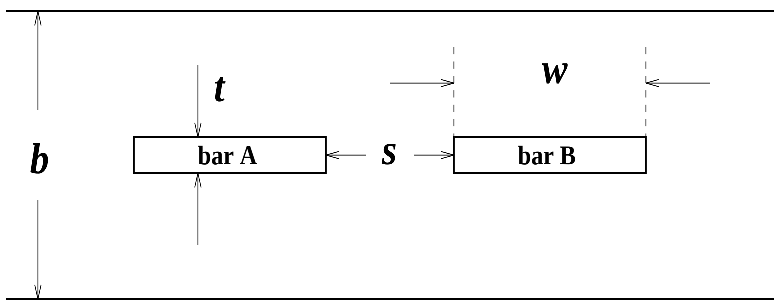

Figure 3: Cross section through parallel rectangular bars between ground planes, showing relevant dimensions.

The configuration for consideration is that shown in figure 3. Two rectangular bars lie parallel in the plane perpendicular to the surface of the paper. They are of equal thickness (t) and situated mid-way between ground planes spaced apart by b, which are in turn parallel with the bars.

The capacitance associated with each bar can be segregated into two portions: that to the ground planes, equivalent to C_A or C_B, and the mutual capacitance ΔC equivalent to capacitance C_{AB} of figure 2.

impedance by simple inversion. The definitions of characteristic impedance require that the other lines of an ensemble be driven with equal currents, not voltages, and this leads to values which in the case of non-identical coupled lines are not the reciprocals of the characteristic admittances.

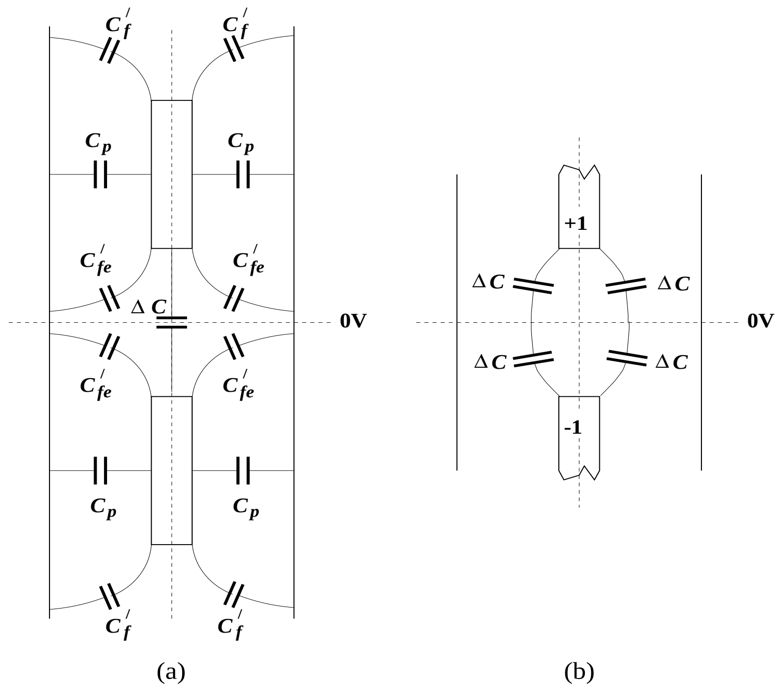

Figure 4: (a) The coupling and to-ground capacitances associated with the structure to be analysed and (b) the coupling capacitance replaced by four equal capacitances to simplify analysis.

Consider first the pair of identical bars shown in figures 3 and 4. These bars are assumed initially to have a width, w, sufficiently large that the fields associated with each end of a bar may be considered individually, *i.e.* the intervening field is sensibly uniform over a range. The capacitance to ground for each bar can be split into three pairs of capacitances. One pair comprises the idealised parallel plate capacitance that would exist from each face to ground (C_p). The second pair (C'_{fe}) is associated with the corners adjacent to the coupling gap; these capacitances represent the perturbing values arising from fringing effects and each is equal to the difference between the actual capacitance to earth and C_p. The third pair of capacitances (C'_f) arises similarly from the fringing effects at the two corners of each bar remote from the coupling gap. Thus, the total capacitance to earth from a bar is given by

$$2(C_p + C'_{fe} + C'_f) \tag{4}$$

Under even-mode conditions of excitation, the capacitance to earth is designated C_{0e} and is equal to the capacitance defined above.

Under odd-mode conditions of excitation, the apparent capacitance from either bar to ground is $2(C_p + C'_{fe} + C'_f + \Delta C)$. In this condition, the plane between the bars is at zero potential, and may be replaced by a zero potential conducting wall for the purpose of analysis. The coupling capacitance ΔC may then be split into four components each equal to the original

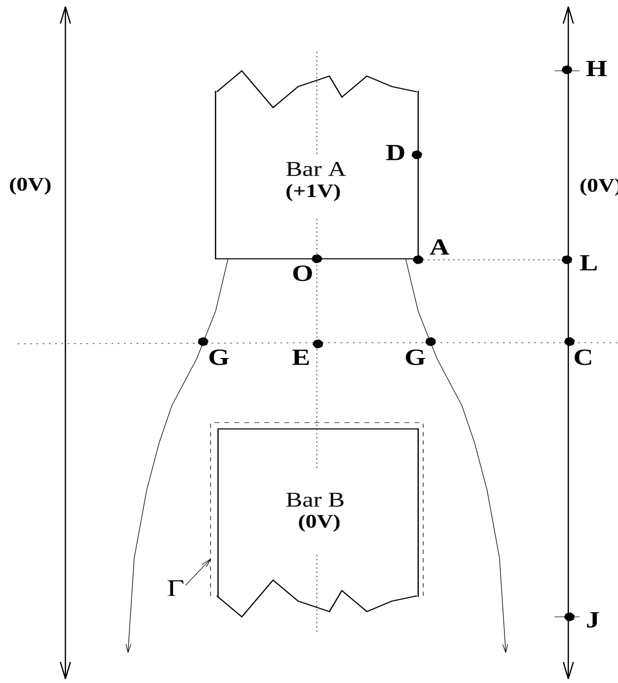

Figure 5: The definition of the separatrices and points G.

coupling capacitance as shown in figure 4b.

The capacitances mentioned above can be defined in precise terms. Referring to figure 5, we see that

$$\frac{\Delta C}{\epsilon} \triangleq \oint_{\Gamma} \vec{E} \cdot \mathrm{d}\vec{S} = \int_{E}^{G} \vec{E} \cdot \mathrm{d}\vec{S} \tag{5}$$

$$\frac{C_p(\mathrm{LH})}{\epsilon} \triangleq \frac{\mathrm{LH}}{\mathrm{AL}} \tag{6}$$

$$\frac{C'_{fe}}{\epsilon} \triangleq \lim_{\mathrm{H}\to\infty} \left\{ \int_{\mathrm{C}}^{\mathrm{H}} \vec{E} \cdot \mathrm{d}\vec{S} - C_p(\mathrm{LH}) \right\} + \lim_{\mathrm{J}\to-\infty} \int_{\mathrm{J}}^{\mathrm{C}} \vec{E} \cdot \mathrm{d}\vec{S} \tag{7}$$

The capacitance C'_{fe} is a function of both ground plane spacing and bar separation; clearly C'_f is equal to the limiting value of C'_{fe} as the bar separation approaches infinity. The lines through G in figure 5 define separatrices, dividing the field from bar A into two portions, one going to ground and one going to bar B. The points G are those at which the separatrices cross the central plane between the bars.

The configuration to be analysed is shown in figure 6. This is in odd-mode excitation and represents the cross section through one bar structure with the zero potential wall in place. The origin is point O and the bar is assumed to extend to B at $j\infty$. The co-ordinates of points in figure 6 are normalised with respect to $b/2$, the half-ground-plane spacing. Point A is thus $(a, j0)$ where

$$a = (t/2)/(b/2) = t/b \tag{8}$$

Similarly D is at (a, jd), with $d = AD/(b/2)$, B is at $(x, j\infty)$ (where $a \leq x \leq 1$), C is at $(1, -js/b)$, G is at $(EG/(b/2), -js/b)$ and E is at $(0, -js/b)$.

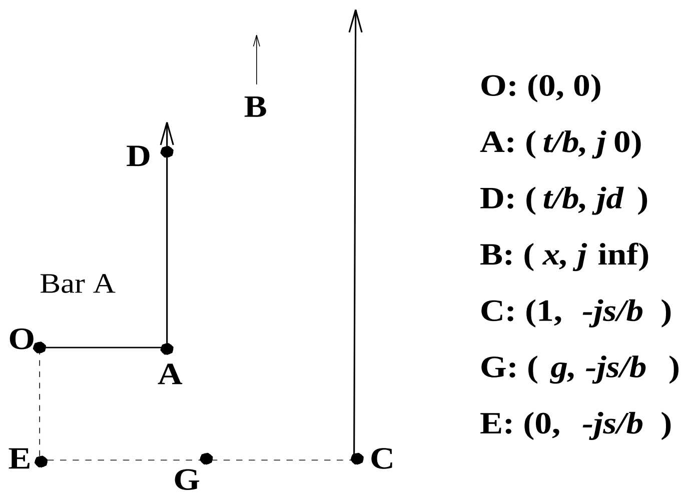

Figure 6: A cross section of the configuration to be analysed, with normalised dimensions.

Since the system is symmetrical, it is only necessary to analyse the quarter segment shown in figure 6. The capacitances are evaluated by conformal transformation. The solution in [2] for ΔC in closed form involves elliptic functions. The present work required the numerical computation of integrals which are amenable to computer evaluation. The present analysis then falls into two parts: the evaluation of the relationship between bar dimensions and coupling capacitance, and the evaluation of the fringing field component and its effective perturbing distance.

The first conformal transformation to be employed is of a Schwarz-Christoffel type given by

$$\frac{\mathrm{d}z}{\mathrm{d}z'} = \frac{k'}{z'^2 - 1}\sqrt{\frac{z'^2 - a'^2}{z'^2 - c'^2}} \tag{9}$$

This transforms the interior of the shape OADBCGEO of figure 6 into the lower right quarter of the z' plane as shown in figure 7a, with the points O′, A′, D′, B′, C′, G′ and E′ corresponding with their unprimed counterparts in the z plane. The transformation has located the point O′ at $(0, j0)$, B′ at $(1, j0)$, G′ at $(g', j0)$ and E′ at $(\infty, j0)$. The points A′, C′, D′ are at $(a', j0)$, $(c', j0)$ and $(d', j0)$ respectively and $\infty > g' > c' > 1 > d' > a' > 0$. It should be noted that the left-hand side of the configuration in the z plane (not shown) is transformed symmetrically into the left hand side of the z' plane.

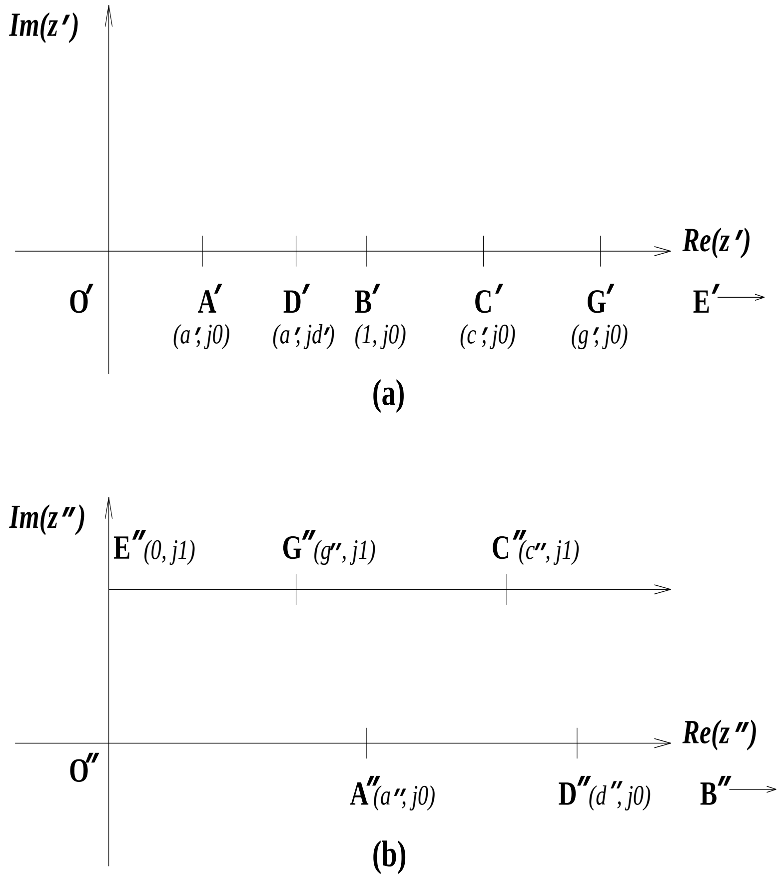

Figure 7: The transformations employed in the calculation: (a) the primed plane, (b) the double-primed plane.

At z'=0, $\mathrm{d}z/\mathrm{d}z'$ is equal to $-k'a'/c'$, and hence k' is essentially negative. It is convenient to re-arrange the transformation with $k = -k'$, so that k is essentially positive and the transformation becomes

$$\frac{\mathrm{d}z}{\mathrm{d}z'} = kF\left(z'\right) = \frac{k}{1 - z'^2}\sqrt{\frac{z'^2 - a'^2}{z'^2 - c'^2}} \tag{10}$$

This initial transformation leads to a second which produces a form from which the capacitances can be readily deduced. This second transformation is

$$z'' = \frac{1}{\pi}\ln\left(\frac{1+z'}{1-z'}\right) \tag{11}$$

As shown in figure 7b, this latter transformation converts that part of the $\Re z'$ axis between O′ and B′ into the positive half axis of the z'' plane with points 0′ $(0, j0)$ and B′ $(1, j0)$ in the z' plane becoming corresponding points O″ $(0, j0)$ and B″ (∞, jy), $0 \leq y \leq 1$, in the z''

plane. That portion of the $\Re z'$ axis between B$'$ $(1, j0)$ and E$'$ $(\infty, j0)$ becomes the segment between B$''$ (∞, jy) and E$''$ $(0, j1)$ in the z'' plane. Negative points on the $\Re z'$ axis appear in corresponding negative positions on the z'' plane.

It is apparent that the transformation from the z plane to the z'' plane is given directly by

$$\frac{\mathrm{d}z}{\mathrm{d}z''} = \frac{k\pi}{2} G(z'') \tag{12}$$

where

$$G(z'') = \left\{ \frac{\left(1 - e^{-\pi z''}\right)^2 - a'^2 \left(1 + e^{-\pi z''}\right)^2}{\left(1 - e^{-\pi z''}\right)^2 - c'^2 \left(1 + e^{-\pi z''}\right)^2} \right\}^{1/2} \tag{13}$$

Clearly if the inner bar OAB is held at a given potential, with the ground planes and EC at zero, then in the z'' plane the line segment having $jy'' = j0$ will be at the given potential, whilst the segment having $jy'' = j1$ will be at zero. The surfaces so defined now form a parallel plate capacitor. The essential property of conformal transformations means that the coupling capacitance ΔC in figure 4 is equal to that in figure 5 between EG and bar A; it is also equal to that in figure 7b between E$''$G$''$ and the $\Re z''$ axis, *i.e.* $\Delta C = g''\epsilon$, where ϵ is the dielectric constant of the medium. From the appendix of [2], it can be deduced that

$$\frac{\Delta C}{\epsilon} = \frac{1}{\pi} \ln \left(\frac{c'^2}{c'^2 - 1} \right) = g'' \tag{14}$$

This expression relates g'' to c'.

In normal usage the capacitance ΔC is known and the major problem is to determine the corresponding dimensions in the z plane. At this stage therefore, the values of b, the ground plane spacing, and t, the centre bar thickness, are assumed to be known also, with s to be determined.

The process of derivation is most readily followed by inverting the transformations in sequence. Given $\Delta C/\epsilon$ $(= g'')$ we may find c' from

$$c' = \sqrt{\frac{1}{1 - e^{-\pi \Delta C/\epsilon}}} \tag{15}$$

The next stage is to consider the situation in the z' plane. Here the transformed polygon of the z plane now lies along the $\Re z'$ axis, with the interior lying below the axis. A contour integral can be delineated following the path from E$'$ $(\infty, j0)$ to O$'$ $(0, j0)$, O$'$ to $(0, -j\infty)$, and an infinite arc from $(0, -j\infty)$ to E$'$ $(\infty, j0)$. There is a pole on the $\Re z'$ axis at $\Re z' = 1$, and the contour must be indented to pass below this. The integral, which by Cauchy's theorem is zero, may be expressed as the sum of several parts *i.e.*

$$k\left[\int_{\infty}^{c'}+\int_{c'}^{1+\varepsilon}+\int_{1-\varepsilon}^{a'}+\int_{a'}^{0}+\int_{0}^{-j\infty}+\int_{-j\infty}^{\infty}\right]F(z')\mathrm{d}z'+J=0 \tag{16}$$

where J is the residue from the indentation at $\Re z'=1$. The integrals involving ε are taken in the limit $\varepsilon\to 0$.

We first deal with

$$\int_{-j\infty}^{\infty}F(z')\mathrm{d}z' \tag{17}$$

On this infinite arc the integrand is approximated by $-1/z'^2$. Replacing z' by $Re^{j\theta}$, where $R\to\infty$, the integral becomes

$$\lim_{R\to\infty}\int_{-\pi/2}^{0}-\frac{1}{R^2e^{2j\theta}}jRe^{j\theta}\mathrm{d}\theta=\lim_{R\to\infty}\int_{-\pi/2}^{0}-j\frac{e^{-j\theta}}{R}\mathrm{d}\theta=0 \tag{18}$$

The term J is the integral around the indentation at $\Re z'=1$ and may be evaluated by the substitution $z'=1+re^{j\theta}$ and letting $r\to 0$, or else by evaluating the residue by conventional means. Either way

$$J=-k\frac{\pi}{2}\sqrt{\frac{1-a'^2}{c'^2-1}} \tag{19}$$

Examining the other integrals individually, we can make the following identifications

$$k\int_{\infty}^{c'}F(z')\mathrm{d}z'=1 \tag{20}$$

In the z plane this is the transition from E to C normalised with respect to $b/2$.

The integrals

$$k\int_{c'}^{1+\varepsilon}F(z')\mathrm{d}z'+k\int_{1-\varepsilon}^{a'}F(z')\mathrm{d}z'+J \tag{21}$$

taken together have a (Cauchy) principal value equal to $-(b-t)/b+js/b$ corresponding with the transition from C to A in the z plane. Identifying real and imaginary parts gives

$$\frac{b-t}{b}=k\frac{\pi}{2}\sqrt{\frac{1-a'^2}{c'^2-1}} \tag{22}$$

and

$$\frac{js}{b}=k\int_{c'}^{1+\varepsilon}F(z')\mathrm{d}z'+k\int_{1-\varepsilon}^{a'}F(z')\mathrm{d}z' \tag{23}$$

The remaining integrals are given by

$$-\frac{t}{b} = k\int_{a'}^{0} F(z')\mathrm{d}z' \tag{24}$$

and

$$-\frac{js}{b} = k\int_{0}^{-j\infty} F(z')\mathrm{d}z' \tag{25}$$

The summation of the integrals and residue are zero, as required.

We can now re-state the problem usually to be solved. It is: given the values of ΔC, b and t, determine the value of s.

From the equation 22,

$$k = \frac{2}{\pi}\frac{b-t}{b}\sqrt{\frac{c'^2-1}{1-a'^2}} \tag{26}$$

and hence, using equation 24,

$$\frac{t}{b} = \frac{2}{\pi}\frac{b-t}{b}\sqrt{\frac{c'^2-1}{1-a'^2}}\int_0^{a'}\sqrt{\frac{z'^2-a'^2}{z'^2-c'^2}}\frac{\mathrm{d}z'}{1-z'^2} \tag{27}$$

In this equation, c', b and t are all known and the equation can be solved for a' by the numerical technique discussed in Appendix I. Given a', the value of k immediately follows. The value of s/b can now be found from equations 22 and 25 which give

$$\frac{-s}{b} = \frac{2}{\pi}\frac{b-t}{b}\sqrt{\frac{c'^2-1}{1-a'^2}}\int_0^{\infty}\sqrt{\frac{x^2+a'^2}{x^2+c'^2}}\frac{\mathrm{d}x}{1+x^2} \tag{28}$$

Values of $\Delta C/\epsilon$ and C'_{fe}/ϵ have been calculated in accordance with the above and are plotted in figure 8. The curves are identical to those of figure (3) of [2].

3. Determination of fringing capacitance

In discussing the equivalent circuit of the pair of coupled bars, we referred to the fringing capacitances C'_{fe}. It is necessary to know the value of this when a filter design is in hand. The fringing capacitance is defined as the difference between the actual capacitance between one side face of the centre bar and ground, and the idealised parallel plate capacitance, when the bar side dimension tends to infinity . The parallel plate capacitance between A and D in figure 5 is $C_p(d) = 2\epsilon \mathrm{AD}/(b-t)$. The total capacitance to ground, considering one side and the end of the bar adjacent to the coupling gap end of the bar only, is $C_p(d) + C'_{fe}$. We can determine the value of C'_{fe} by noting that

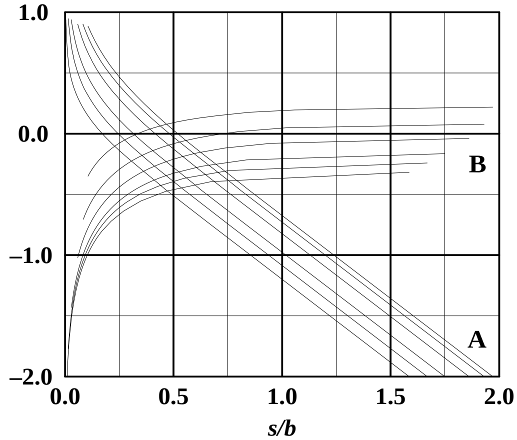

Figure 8: (A) $\log_{10}(\Delta C/\epsilon)$ and (B) $\log_{10}(C'_{fe}/\epsilon)$ *versus* s/b for $t/b = 0.0$, 0.1, 0.2, 0.4, 0.6 and 0.8 (from bottom upwards).

$$\frac{C'_{fe}}{\epsilon} = \lim_{d'' \to \infty} d'' - \frac{C_p(d)}{\epsilon} - \frac{\Delta C}{\epsilon} \tag{29}$$

Now

$$\frac{C_p(d)}{\epsilon} = \frac{2\mathrm{AD}}{b-t} = \frac{db}{b-t} \tag{30}$$

and

$$jd = k\frac{\pi}{2}\int_{a''}^{d''} G(z'')\mathrm{d}z'' \tag{31}$$

whence

$$\frac{C'_{fe}}{\epsilon} = a'' - \frac{\Delta C}{\epsilon} + \lim_{d'' \to \infty}\int_{a''}^{d''} 1 + j\sqrt{\frac{c'^2 - 1}{1 - a'^2}}G(z'')\mathrm{d}z'' \tag{32}$$

where $a'' = 1/\pi \ln(1 + a')/(1 - a')$. Thus given $\Delta C/\epsilon$, t, and b, it is possible to find s, C'_{fe}/ϵ. The value of C_f/ϵ, the fringing capacitance associated with a bar end where there is no neighbouring bar, is equal to the limiting value of C'_{fe}/ϵ when the inter-bar spacing s approaches infinity.

The expression for C'_{fe}/ϵ has been left with d'' finite in order to explore the limits of validity. As d'' increases the value of C'_{fe}/ϵ rapidly approaches a limit. Physically, this corresponds with the onset of field uniformity in the gap. In order to determine a suitable value of d at which it may be deemed that the field has reached uniformity, we may calculate the value of d at which the value of C'_{fe}/ϵ has reached a given fraction of its final value. Values of $d_{99\%}$ (at

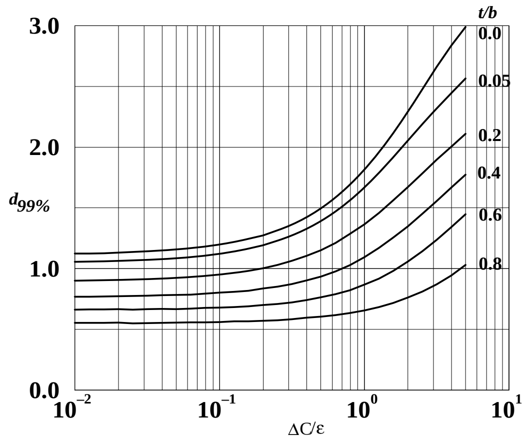

Figure 9: Minimum bar dimension for which C'_{fe} has reached 99% of its final value

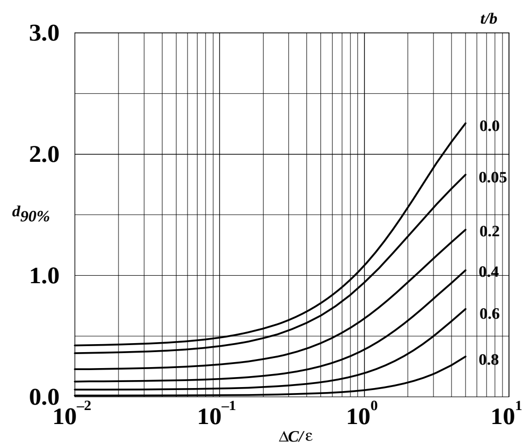

Figure 10: Minimum bar dimension for which C'_{fe} has reached 90% of its final value

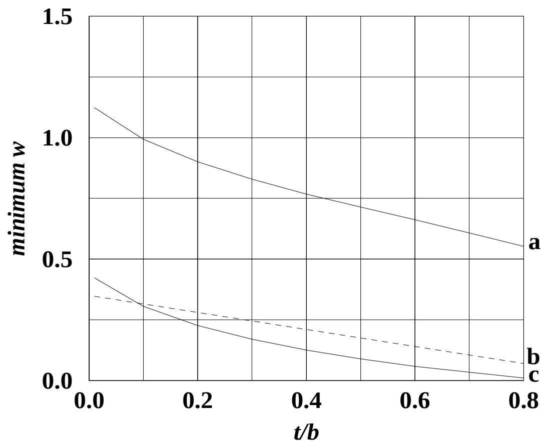

Figure 11: Minimum bar width for weakly coupled bars as a function of t/b with $\Delta C/\epsilon = 0.01$. (a) $w_{99\%}$, (b) $0.35(1 - t/b)$ and (c) $w_{90\%}$. N.B. widths normalised with respect to b.

which C'_{fe} has reached 0.99 of its final value) and $d_{90\%}$ (C'_{fe} has reached 0.9 of its final value) have been calculated and plotted in figures 9 and 10, for a range of values of $\Delta C/\epsilon$ with t/b as parameter.

If we now assume that $d_{99\%}$ (or $d_{90\%}$) represents a distance sufficiently far from the bar end that the field is effectively normal, then we may postulate that a bar of width equal to twice this distance has the minimum width for which the foregoing treatment is applicable. If the bar width w is normalised with respect to b, then $w_{99\%}$ is numerically equal to $d_{99\%}$ *etc.*

Whilst figures 9 and 10 show that values of minimum bar dimension (and hence minimum width) vary appreciably with t/b and $\Delta C/\epsilon$, this is not very helpful in deriving rules of thumb for practical applications. Such a rule may be derived by replotting the data for a bar with very weak coupling (*e.g.* $\Delta C/\epsilon = 0.01$) to its neighbours: $w_{99\%}$ and $w_{90\%}$ are plotted as a function of t/b in figure 11. From this it can be seen that for $w_{99\%}$, minimum normalised width varies between 1.2 and 0.6 whilst for $w_{90\%}$ it varies between 0.5 and 0.05. The only other criterion for minimal interaction between fringing fields known to the present authors is that given in [3] as $w > 0.35(1 - t/b)$. This has been plotted in figure 11, from which it would appear to be of somewhat limited value.

APPENDIX

Details of the numerical calculation of s/b for a given $\Delta C/\epsilon$, t and b

Step 1 Given $\Delta C/\epsilon$, calculate c', given by

$$c' = \frac{1}{\sqrt{1 - e^{-\pi \Delta C/\epsilon}}}$$

Step 2 Find a' such that[2]

$$I(a') \triangleq \frac{2}{\pi}\sqrt{\frac{c'^2 - 1}{1 - a'^2}} \int_0^{a'} \sqrt{\frac{a'^2 - x^2}{c'^2 - x^2}} \frac{\mathrm{d}x}{1 - x^2} = \frac{t}{b - t}$$

This is best done in two parts. The first part is to find n such that

$$I(1 - 10^{-n}) < \frac{t}{b - t} \leq I(1 - 10^{-n-1})$$

where $n = 0, 1, 2 \ldots$ This initial bracketing is done because the value of $1 - a'$ varies over several orders of magnitude for $\Delta C/\epsilon$ in the range of interest.
The second part is to use the bisection method [4] with the initial range $1 - 10^{-n}$ to $1 - 10^{-n-1}$ to find a more accurate value of a'. The criterion used for stopping bisection was that

$$|I(a') - t/(b - t)| \leq 2 \times 10^{-5}$$

Such accuracy was always achieved within 50 iterations.

Step 3 This value of a' allows us to find s from

$$s = -(b - t)\frac{2}{\pi}\sqrt{\frac{c'^2 - 1}{1 - a'^2}} \int_0^{\infty} \sqrt{\frac{a'^2 + x^2}{c'^2 + x^2}} \frac{\mathrm{d}x}{1 + x^2}$$

from which s/b follows.

Note: The integrand of $I(a')$ behaves in such a way that necessitates the use of rather careful numerical integration since a' is a branch point. An adaptive integration routine using the Gauss 10-point and the Kronrod 21-point rules, as implemented in the NAG library, was found to be suitable.

[2] If $t = 0$, $a' = 0$ and we can miss out step 2.

References

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