WEBSTER'S HORN EQUATION REVISITED[*]

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Abstract. The problem of low-frequency sound propagation in slowly varying ducts is systematically analyzed as a perturbation problem of slow variation. Webster's horn equation and variants in bent ducts, in ducts with nonuniform soundspeed, and in ducts with irrotational mean flow, with and without lining, are derived, and the entrance/exit plane boundary layer is given. It is shown why a varying lined duct in general does not have an (acoustic) solution.

Key words. Webster's horn equation, duct acoustics, sound propagation in lined ducts with mean flow, perturbation methods, method of slow variation

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1. Introduction. Sound of long wavelength, propagating in ducts of varying diameter like horns, is suitably described by an approximate equation, known as Webster's horn equation or just Webster's equation. This is an ordinary differential equation in the axial coordinate, and therefore forms a significant simplification of the problem [1, 2, 3].

The usual derivation is based on the assumption of a crosswise uniform acoustic pressure field, such that, by averaging over a duct cross section, the spatial dimensions of the problem are reduced from three to one.

Although it shows a remarkable evidence of ingenuity and physical insight, this derivation is mathematically unsatisfying. It is not clear (i) what exactly is the small parameter underlying the approximation, (ii) why the pressure may be assumed to be uniform, (iii) what the error is of the approximation, (iv) what the conditions are on the duct geometry and on the frequency of the field, (v) how to generalize to similar problems, (vi) how to generate higher order corrections, and (vii) what happens near the source or duct entrance or exit plane.

An asymptotically systematic derivation of the three-dimensional (3D) classic problem was given by Lesser and Crighton [4], extending the derivation of Lesser and Lewis in [5, 6]. They also showed for a number of 2D configurations how abrupt changes of the geometry (open end, slit in the wall) can be incorporated as boundary layer regions in a setting of matched asymptotic expansion. Their approach, based on introducing different longitudinal and lateral scales, is a special case of the method of slow variation put forward by Van Dyke [7]. Although only an asymptotically sound derivation is able to indicate the range of validity and the order of the error of the approximation, we found in the literature no variants of this problem (e.g., with mean flow [8, 9, 10, 11, 12]) that strictly follow that approach.

Particularly interesting would be an investigation of the related problems of lined ducts without and with flow, as this would form a natural long wavelength closure of the multiple scales theory of sound propagation in slowly varying ducts [13, 14, 15, 16].

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Another problem of practical interest that is directly related to a systematic set-up is the entrance problem for a 3D duct of arbitrary cross section. The structure of the boundary layer was indicated by Lesser and Crighton [4], but they gave explicit examples only for 2D geometries.

All in all, while the problem of long wave sound propagation in slowly varying ducts, in various generalizations, is practically important, it still has a lot of open ends.

We will consider various cases in detail. First, we show how a systematic approach, known as the method of slow variation coupled with ideas of matched asymptotic expansions, leads to the classic Webster equation for hard-walled ducts with the entrance boundary layer. The small parameter ε is equal to the Helmholtz number, the ratio between a typical wavelength and the duct diameter, while a typical length scale of duct variation is of the same order of magnitude as the wavelength. Using similar results for the related problem of heat conduction [17], this entrance problem will be solved explicitly. It leads via matching conditions to conclusions about the way that the $\mathcal{O}(1)$ duct field error ($\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\varepsilon^2)$) depends on the source.

Then we will show that our problem is not essentially different in other coordinate systems (like spherical coordinates), although special coordinates may be helpful in obtaining a more efficient approximation. Curved ducts, with a curvature radius of no more than the typical length scale of diameter variation, are shown to still produce the same equation.

The same type of analysis can be applied to ducts with lined walls of, say, impedance Z. It is found that for $Z = \mathcal{O}(1)$ only the trivial solution exists, while for $Z = \mathcal{O}(\varepsilon)$ there are only nontrivial solutions possible for certain geometry-dependent values of the wall impedance. As these impedance values vary along the duct, there are in general no solutions possible for the full duct. A subtle functional analytic result is used, due to Professor Jan de Graaf (TU Eindhoven), which is not available in the literature; therefore, Prof. de Graaf was kind enough to attach his derivation as an appendix to this paper.

We continue with more general analyses of the problem in a stagnant medium with slowly varying sound speed, and of sound in an irrotational isentropic mean flow, leading to generalized forms of Webster's horn equation.

We finish with the same problem with mean flow but now extended to ducts with lined walls. Using a recent result obtained for the related problem for high-frequency sound propagation in lined flow ducts [16], we are able to show for $Z = \mathcal{O}(1)$ that also here only a special hydrodynamic (nonacoustic) wave is possible.

2. The physical models.

2.1. The equations. In the acoustic realm of a perfect gas that we will consider, we have for pressure $\tilde{p}$, velocity $\tilde{\boldsymbol{v}}$, density $\tilde{\rho}$, entropy $\tilde{s}$, and soundspeed $\tilde{c}$

(1)
$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -\tilde{\rho}\nabla\boldsymbol{\cdot}\tilde{\boldsymbol{v}}, \quad \tilde{\rho}\frac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t} = -\nabla\tilde{p}, \quad \frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = 0,$$
$$\mathrm{d}\tilde{s} = C_V\frac{\mathrm{d}\tilde{p}}{\tilde{p}} - C_P\frac{\mathrm{d}\tilde{\rho}}{\tilde{\rho}}, \quad \tilde{c}^2 = \frac{\gamma\tilde{p}}{\tilde{\rho}}, \quad \gamma = \frac{C_P}{C_V},$$

where γ, C_P, and C_V are gas constants. When the flow originates from a thermo-dynamically uniform state and consists of a stationary mean flow, with unsteady

time-harmonic perturbations of frequency ω given, in the usual complex notation, by

(2)
$$\tilde{\boldsymbol{v}} = \boldsymbol{V} + \mathrm{Re}(\boldsymbol{v}\,\mathrm{e}^{\mathrm{i}\omega t}), \quad \tilde{p} = P + \mathrm{Re}(p\,\mathrm{e}^{\mathrm{i}\omega t}), \quad \tilde{\rho} = D + \mathrm{Re}(\rho\,\mathrm{e}^{\mathrm{i}\omega t}), \quad \tilde{s} = S + \mathrm{Re}(s\,\mathrm{e}^{\mathrm{i}\omega t})$$

($\omega > 0$), we obtain for the mean flow, upon linearization for small amplitude,

(3)
$$\nabla \cdot (D\boldsymbol{V}) = 0, \quad D(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\nabla P,$$
$$(\boldsymbol{V} \cdot \nabla)S = 0, \quad S = C_V \log P - C_P \log D, \quad C^2 = \frac{\gamma P}{D},$$

and for the perturbations

(4a)
$$\mathrm{i}\omega\rho + \nabla \cdot (\boldsymbol{V}\rho + \boldsymbol{v}D) = 0,$$

(4b)
$$D(\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla)\boldsymbol{v} + D(\boldsymbol{v} \cdot \nabla)\boldsymbol{V} + \rho(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\nabla p,$$

(4c)
$$(\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla)s + \boldsymbol{v} \cdot \nabla S = 0,$$

while

(4d)
$$s = \frac{C_V}{P}p - \frac{C_P}{D}\rho = \frac{C_V}{P}(p - C^2\rho).$$

Without mean flow, such that $\boldsymbol{V} = \nabla P = 0$, the equations may be reduced to (see section 8)

(5)
$$\nabla \cdot (C^2 \nabla p) + \omega^2 p = 0.$$

If, in addition, the ambient medium is uniform, with a constant soundspeed C and density D, the acoustic field becomes isentropic and irrotational, and we may introduce a potential $\boldsymbol{v} = \nabla\phi$. Furthermore, (5) reduces to the Helmholtz equation. After introducing the free field wave number $k = \omega/C$, we have (see sections 3, 4, 6, 7)

(6)
$$\nabla^2 \phi + k^2 \phi = 0.$$

If the original flow field $\tilde{\boldsymbol{v}}$ is irrotational and isentropic everywhere (homentropic), we can introduce a potential for the velocity, where $\tilde{\boldsymbol{v}} = \nabla\tilde{\phi}$, and express $\tilde{p}$ as a function of $\tilde{\rho}$ only, such that we can integrate the momentum equation (Bernoulli's law, with constant E) to obtain for the mean flow

(7)
$$\tfrac{1}{2}V^2 + \frac{C^2}{\gamma - 1} = E, \quad \nabla \cdot (D\boldsymbol{V}) = 0, \quad \frac{P}{D^\gamma} = \text{constant},$$

and for the acoustic perturbations

(8)
$$(\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla)\rho + \rho\nabla \cdot \boldsymbol{V} + \nabla \cdot (D\nabla\phi) = 0, \quad D(\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla)\phi + p = 0, \quad p = C^2\rho.$$

These last equations are further simplified (eliminate p and ρ and use the fact that $\nabla \cdot (D\boldsymbol{V}) = 0$) to the rather general convected wave equation (see section 9)

(9)
$$D^{-1}\nabla \cdot (D\nabla\phi) - (\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla)\left[C^{-2}(\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla)\phi\right] = 0.$$
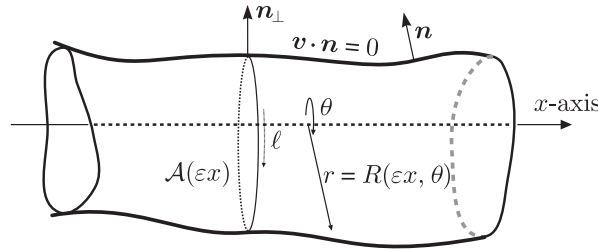
FIG. 1. *Sketch of geometry.*

2.2. Nondimensionalization. Without further change of notation, we will assume throughout this paper that the problem is made dimensionless: lengths on a typical duct radius, time on typical sound speed / typical duct radius, etc.

2.3. The geometry. The domain of interest consists of a duct $\mathcal{V}$ of arbitrary cross section, slowly varying in axial direction (see Figure 1). For definiteness, it is given by the function Σ in cylindrical coordinates, as follows:

$$(10) \qquad\qquad \Sigma(X, r, \theta) = r - R(X, \theta) \leq 0,$$

where $X = \varepsilon x \geqslant 0$ is a so-called slow variable, while ε is small. A cross section $\mathcal{A}(X)$ at axial position X has surface area $A(X)$. Whenever relevant,[1] we assume lengths made dimensionless such that

$$A(0) = 1.$$

At the duct surface $\Sigma = 0$, the gradient $\nabla\Sigma$ is a vector normal to the surface (i.e., $\nabla\Sigma \propto \boldsymbol{n}$), while the transverse gradient $\nabla_\perp \Sigma$,

$$(11) \qquad\qquad \nabla_\perp = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \text{with} \ \ \nabla_\perp \Sigma = \boldsymbol{e}_r - \boldsymbol{e}_\theta \frac{1}{r} R_\theta,$$

(where R_θ denotes the partial derivative of R to θ) is directed in the plane of a cross section $\mathcal{A}(X)$ and normal to the duct circumference $\partial\mathcal{A}$. Thus if $\boldsymbol{n}_\perp$ is the component of the surface normal vector $\boldsymbol{n}$ in the plane of a cross section, we have $\nabla_\perp \Sigma \propto \boldsymbol{n}_\perp$.

2.4. Frequency. The frequencies considered are low, such that the corresponding typical wave number is of the same order of magnitude as the length scale of the duct variations, i.e., dimensionless $\mathcal{O}(\varepsilon^{-1})$. In order to quantify this, we will rescale $k = \varepsilon\kappa$ and $\omega = \varepsilon\Omega$.

3. The classical problem.

3.1. Equations and boundary conditions. The duct is semi-infinite and hard-walled. The solution is determined by a source at entrance plane $x = 0$, and radiation conditions for $x \to \infty$. Other conditions, like a reflecting impedance plane at some exit plane $x = L$ (e.g., modeling a radiating open end [5] or a slit in the wall [4]), are also possible, but they do not essentially alter the present analysis.

Inside $\mathcal{V}$ we have for acoustic potential ϕ (see (6))

$$(12) \qquad \nabla^2 \phi + \varepsilon^2 \kappa^2 \phi = 0 \ \ \text{if} \ \ \boldsymbol{x} \in \mathcal{V}, \ \ \text{with} \ \ \nabla\phi \cdot \boldsymbol{n} = 0 \ \ \text{at} \ \ \boldsymbol{x} \in \partial\mathcal{V}.$$

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[1] In particular, in section 4.

At the entrance interface $x = 0$ we have a suitable boundary condition, say,

$$(13) \qquad\qquad \phi(0, r, \theta) = F(r, \theta).$$

The boundary condition of hard walls at $r = R(X, \theta)$ may be given by

$$(14) \qquad\qquad \nabla_\perp \phi \cdot \nabla_\perp \Sigma = \phi_r - \frac{R_\theta}{R^2} \phi_\theta = \varepsilon R_X \phi_x.$$

Except for the immediate neighborhood of the entrance plane, the typical axial variations of the acoustic field scale on the slow variable X, so we rewrite the equations and boundary conditions as

$$(15) \qquad\qquad \varepsilon^2 \phi_{XX} + \nabla_\perp^2 \phi + \varepsilon^2 \kappa^2 \phi = 0,$$
$$\text{with} \ \ \nabla\phi \cdot \nabla\Sigma = -\varepsilon^2 \phi_X R_X + \nabla_\perp \phi \cdot \nabla_\perp \Sigma = 0 \ \ \text{at} \ \ r = R.$$

This rewriting in a slow variable is known as the method of slow variation [7]. Note that this equation has a small parameter multiplied by the highest derivative in the X-direction, suggesting a singular perturbation problem [4, 18, 19, 20] with boundary layers in X.

3.2. Asymptotic analysis: Outer solution. The following outer solution analysis will largely follow Lesser and Crighton [4], but we will give it in some detail for two reasons. First, we will have to define the solution for the inner solution at the entrance boundary layer to be discussed later. Second, it explicates the method of integration along a cross section that will be used in the various other configurations later.

Based on the observation that ε^2 is the only small parameter that occurs, we might be tempted to expand the solution in a Poincaré asymptotic power series in ε^2. However, we will see that this is not exactly true. Depending on the behavior of the solution near the entrance, the correction term should in general be $\mathcal{O}(\varepsilon)$ for matching. Nevertheless, the leading and first order equations will be equivalent. With the assumed Poincaré expansion of ϕ, expressed in X,

$$(16) \qquad \phi(X, r, \theta; \varepsilon) = \phi_0(X, r, \theta) + \varepsilon \phi_1(X, r, \theta) + \varepsilon^2 \phi_2(X, r, \theta) + \cdots,$$

we obtain to leading order

$$(17) \qquad\qquad \nabla_\perp^2 \phi_0 = 0, \ \ \text{with} \ \ \nabla_\perp \phi_0 \cdot \boldsymbol{n}_\perp = 0 \ \ \text{at} \ \ r = R,$$

with a solution $\phi_0 = 0$. As the solution of a Neumann problem is unique up to a constant, $\phi_0 = \phi_0(X)$, a function to be determined. To first order we have

$$(18) \qquad\qquad \nabla_\perp^2 \phi_1 = 0, \ \ \text{with} \ \ \nabla_\perp \phi_1 \cdot \boldsymbol{n}_\perp = 0 \ \ \text{at} \ \ r = R,$$

also with a constant solution, and so $\phi_1 = \phi_1(X)$, a function to be determined. To second order we now have

$$(19) \ \ \nabla_\perp^2 \phi_2 + \phi_{0,XX} + \kappa^2 \phi_0 = 0, \qquad \text{with} \ \ \nabla_\perp \phi_2 \cdot \boldsymbol{n}_\perp = \phi_{0,X} \frac{R R_X}{\sqrt{R^2 + R_\theta^2}} \ \ \text{at} \ \ r = R.$$

The assumption (16) that there exists a Poincaré expansion for ϕ, expressed in this slow variable X, is not trivial. (Poincaré expansions are critically dependent on the

variables chosen!) It requires certain solvability conditions for, e.g., ϕ_2, yielding an equation for ϕ_0. To obtain this, we integrate along a cross section $\mathcal{A}(X)$ and apply Gauss' theorem

$$\iint_{\mathcal{A}} \nabla_\perp^2 \phi_2 \, \mathrm{d}\sigma = \int_{\partial\mathcal{A}} \nabla_\perp \phi_2 \cdot \boldsymbol{n}_\perp \, \mathrm{d}\ell = \int_{\partial\mathcal{A}} \phi_{0,X} \frac{RR_X}{\sqrt{R^2 + R_\theta^2}} \, \mathrm{d}\ell = \cdots.$$

Then we parametrize $\partial\mathcal{A}$ with θ such that $\mathrm{d}\ell = \sqrt{R^2 + R_\theta^2} \, \mathrm{d}\theta$, and we continue

$$(20) \qquad = \int_0^{2\pi} \phi_{0,X} RR_X \, \mathrm{d}\theta = \phi_{0,X} \int_0^{2\pi} RR_X \, \mathrm{d}\theta = \phi_{0,X} A_X.$$

On the other hand, we also have

$$(21) \qquad \iint_{\mathcal{A}} \left[\phi_{0,XX} + \kappa^2 \phi_0 \right] \mathrm{d}\sigma = A\left(\phi_{0,XX} + \kappa^2 \phi_0 \right).$$

Altogether we have for ϕ_0 the equation

$$(22) \qquad A^{-1}\left(A\phi_{0,X} \right)_X + \kappa^2 \phi_0 = 0,$$

which is indeed Webster's horn equation [1, 2] in properly scaled coordinates.

Evidently, the first order solution follows the same pattern and also satisfies

$$(23) \qquad A^{-1}\left(A\phi_{1,X} \right)_X + \kappa^2 \phi_1 = 0.$$

For completeness we note from [21, 22, 23, 24, 3] that Webster's equation can be recast into a more transparent form by the transformation

$$(24) \qquad A(X) = d(X)^2, \quad \phi = d^{-1}\psi,$$

leading to

$$(25) \qquad \psi'' + \left(\kappa^2 - \frac{d''}{d} \right)\psi = 0.$$

Depending on the sign of $\kappa^2 - d''/d$, the solutions behave like propagating or exponentially decaying waves. Elementary solutions are readily found for geometries with $d''/d = m^2$, a constant, yielding Salmon's family of exponential and conical horns [21, 22].

3.3. Boundary conditions in X. The above equation for ϕ_0 and ϕ_1 is of second order, and therefore two boundary conditions are required to determine the solution. For $X \to \infty$ we have the condition of radiation. At $X = 0$ (Figure 2), ϕ_0 and ϕ_1 cannot satisfy the (r, θ)-dependent boundary condition (13). Indeed, as anticipated before, near $x = 0$ there is a boundary layer of $X = \mathcal{O}(\varepsilon)$, i.e., $x = \mathcal{O}(1)$, which determines the (outer) solutions ϕ_0 and ϕ_1 via conditions of matching. This will be considered in the next section.

4. Entrance boundary layer. Near the entrance, for $X = \mathcal{O}(\varepsilon)$, i.e., $x = \mathcal{O}(1)$, we have of course equation (12)

$$(12) \qquad \nabla^2 \phi + \varepsilon^2 \kappa^2 \phi = 0 \ \text{ if } \ \boldsymbol{x} \in \mathcal{V}, \quad \text{with } \ \nabla_\perp \phi \cdot \boldsymbol{n} = 0 \ \text{ at } \ \boldsymbol{x} \in \partial\mathcal{V}.$$
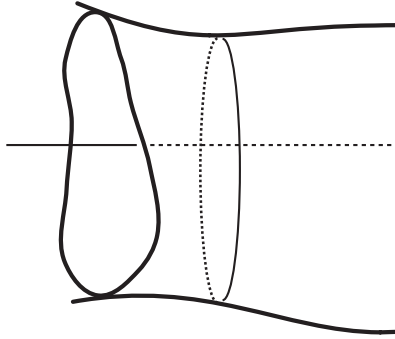
FIG. 2. *The entrance.*

Up to $\mathcal{O}(\varepsilon^2)$, this Helmholtz equation is equivalent to the Laplace equation. Therefore, the boundary layer analysis is essentially similar to that for the heat equation, discussed in Chandra [17]. Expand

$$(26) \qquad \phi(X, r, \theta; \varepsilon) = \Phi_0(x, r, \theta) + \varepsilon\Phi_1(x, r, \theta) + \mathcal{O}(\varepsilon^2)$$

so that we have inside $\mathcal{V}$ to leading and first order,

$$(27a) \qquad\qquad\qquad \mathcal{O}(1): \quad \nabla^2\Phi_0 = 0,$$

$$(27b) \qquad\qquad\qquad \mathcal{O}(\varepsilon): \quad \nabla^2\Phi_1 = 0.$$

At $x = 0$ we have from (13) the initial conditions

$$(28) \qquad\qquad \Phi_0(0, r, \theta) = F(r, \theta), \quad \Phi_1(0, r, \theta) = 0.$$

For $x \to \infty$ conditions of matching with the outer solution $\phi_0 + \varepsilon\phi_1$ apply. For the boundary condition at $r = R$ we have to expand $R(\varepsilon x, \theta)$. Note that for any function f

$$(29)$$
$$f(R(\varepsilon x); \varepsilon) = f(R + \varepsilon x R_X + \mathcal{O}(\varepsilon^2); \varepsilon) = f_0(R) + \varepsilon\big(f_1(R) + x f_{0,r}(R)R_X\big) + \mathcal{O}(\varepsilon^2),$$

where R without any argument denotes the value at $X = 0$. Furthermore, we have

$$(30) \qquad\qquad \frac{R_\theta(X, \theta)}{R^2(X, \theta)} = \frac{R_\theta}{R^2} + \varepsilon x\Big(\frac{R_X}{R^2}\Big)_\theta + \mathcal{O}(\varepsilon^2).$$

Thus at the boundary

$$(31)$$

$$\nabla_\perp\phi \cdot \nabla_\perp\Sigma = \phi_r - \frac{R_\theta}{R^2}\phi_\theta$$

$$= \Phi_{0,r} - \frac{R_\theta}{R^2}\Phi_{0,\theta} + \varepsilon\left[\Phi_{1,r} - \frac{R_\theta}{R^2}\Phi_{1,\theta} + x\Phi_{0,rr}R_X - x\frac{R_\theta}{R^2}R_X\Phi_{0,r\theta} - x\Big(\frac{R_X}{R^2}\Big)_\theta\Phi_{0,\theta}\right]$$

$$= \varepsilon R_X\Phi_{0,x},$$

which means that at $r = R(0, \theta)$ for the leading and first order,

(32a) $\nabla_\perp \Phi_0 \cdot \nabla_\perp \Sigma_0 = \Phi_{0,r} - \dfrac{R_\theta}{R^2} \Phi_{0,\theta} = 0,$

(32b) $\nabla_\perp \Phi_1 \cdot \nabla_\perp \Sigma_0 = \Phi_{1,r} - \dfrac{R_\theta}{R^2} \Phi_{1,\theta}$

$$= R_X \Phi_{0,x} - x \Phi_{0,rr} R_X + x \frac{R_\theta}{R^2} R_X \Phi_{0,r\theta} + x \left(\frac{R_X}{R^2} \right)_\theta \Phi_{0,\theta},$$

where $\Sigma_0 = \Sigma(0, r, \theta)$.

It is important for the subsequent matching to note that the solutions of (27) with (32) are defined only up to a linear term Kx. For Φ_0, however, this would result in terms of $\mathcal{O}(\varepsilon^{-1})$ if $x = \mathcal{O}(\varepsilon^{-1})$, which do not match with an outer solution $\phi_0 = \mathcal{O}(1)$. Therefore, we will not include this extra term. For Φ_1, on the other hand, we will have to retain the possibility, and in the end a linear term $K_1 x$ will be added, where K_1 must be determined by the matching.

From the identity at $r = R$,

(33) $\dfrac{\mathrm{d}}{\mathrm{d}\theta} \Phi_{0,\theta} = \Phi_{0,r\theta} R_\theta + \Phi_{0,\theta\theta},$

and with the defining equation applied at $r = R$ while using relation (32a),

(34) $-\Phi_{0,rr} = \dfrac{1}{R} \Phi_{0,r} + \dfrac{1}{R^2} \Phi_{0,\theta\theta} + \Phi_{0,xx} = \dfrac{R_\theta}{R^3} \Phi_{0,\theta} + \dfrac{1}{R^2} \Phi_{0,\theta\theta} + \Phi_{0,xx},$

it follows that (32b) is equivalent to

(35) $\nabla_\perp \Phi_1 \cdot \nabla_\perp \Sigma_0 = \mathcal{Q}_0(x, \theta)$

$$\stackrel{\text{def}}{=} R_X \Phi_{0,x}\big|_{r=R} + \frac{x}{R} \left\{ R R_X \Phi_{0,xx}\big|_{r=R} + \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{R_X}{R} \Phi_{0,\theta}\big|_{r=R} \right) \right\}.$$

4.1. Leading order. The right-running solution Φ_0 (only nonincreasing exponentials are allowed for matching) may be expressed by the eigenfunction expansion

(36) $\Phi_0(\boldsymbol{x}) = \displaystyle\sum_{n=0}^{\infty} F_n \psi_n(r, \theta) \, \mathrm{e}^{-\lambda_n x},$

where

(37) $\nabla_\perp^2 \psi_n + \lambda_n^2 \psi_n = 0, \quad \nabla_\perp \psi_n \cdot \nabla_\perp \Sigma_0 = 0,$

with $\lambda_0 = 0$, ψ_0 a constant (normalized to 1), the other eigenvalues[2] λ_n real positive, and the eigenfunctions ψ_n real, orthogonal, and assumed normalized. In general these eigenfunctions are to be determined numerically. However, if the duct is cylindrical (i.e., R is independent of θ), we have

(38) $\psi_n(r, \theta) := \psi_{\nu\mu}(r, \theta) = \begin{cases} \dfrac{J_\nu\left(j'_{\nu\mu} r/R \right)}{\sqrt{\dfrac{\pi}{2} \left(1 - \dfrac{\nu^2}{j'_{\nu\mu}{}^2} \right)} R J_\nu(j'_{\nu\mu})} \begin{Bmatrix} \cos\nu\theta \\ \sin\nu\theta \end{Bmatrix} & \text{for } \nu \neq 0, \\[2em] \dfrac{J_0\left(j'_{0\mu} r/R \right)}{\sqrt{\pi} R J_0(j'_{0\mu})} & \text{for } \nu = 0, \end{cases}$

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[2] Strictly speaking, the numbers $-\lambda_n^2$ are the eigenvalues of operator $\nabla_\perp^2$.

where the index n is more practically changed into the double index $(\nu\mu)$. J_ν is the νth order ordinary Bessel function of the first kind [25], and $j'_{\nu\mu}$ is the μth (real-valued, positive) zero of J'_ν. The corresponding eigenvalue is thus $\lambda_n := j'_{\nu\mu}/R$.

The amplitudes are determined from the entrance interface $x = 0$ as follows:

$$(39) \qquad F_n = \iint_{\mathcal{A}(0)} F(r,\theta)\psi_n(r,\theta)\,\mathrm{d}\sigma.$$

Note that, as ψ_n are orthonormal, the axial flux is, to leading order, proportional to the imaginary part of

$$(40) \qquad \int_0^{2\pi}\int_0^R \Phi_0\Phi_{0,x}^* r\,\mathrm{d}r\mathrm{d}\theta = -\sum_{n=1}^\infty \lambda_n|F_n|^2\,\mathrm{e}^{-2\lambda_n x}.$$

As this expression is real, its imaginary part is zero, and thus the axial flux vanishes to leading order. Indeed, the outer solution is a slowly varying function of X, and therefore the flux, proportional to the axial derivative, is $\mathcal{O}(\varepsilon)$.

For $x \to \infty$, the exponential terms in $\Phi_0(\boldsymbol{x})$ vanish and we have

$$(41) \qquad \Phi_0(\boldsymbol{x}) \simeq F_0.$$

4.2. First order. With the found expression for Φ_0, the right-hand side of (35), $\mathcal{Q}_0$, may be written as

$$\mathcal{Q}_0(x,\theta) = \sum_{n=1}^\infty F_n\,\mathrm{e}^{-\lambda_n x}\left[-R_X\lambda_n\psi_n\big|_{r=R} + xR_X\lambda_n^2\psi_n\big|_{r=R} + \frac{x}{R}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_X}{R}\psi_{n,\theta}\big|_{r=R}\right)\right]$$

$$(42) \qquad = R^{-1}\sum_{n=1}^\infty F_n\left[-\lambda_n RR_X\left(x\,\mathrm{e}^{-\lambda_n x}\right)_x\psi_n\big|_{r=R} + x\,\mathrm{e}^{-\lambda_n x}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_X}{R}\psi_{n,\theta}\big|_{r=R}\right)\right].$$

To solve the problem for Φ_1, we introduce a Green's function $G(\boldsymbol{x};\boldsymbol{\xi})$ with $\boldsymbol{x} = (x,r,\theta)$ and $\boldsymbol{\xi} = (\xi,\rho,\eta)$ satisfying

(43)

$$\nabla_\perp^2 G + \frac{\partial^2}{\partial x^2}G = -\delta(\boldsymbol{x}-\boldsymbol{\xi}), \quad \frac{\partial}{\partial n}G = 0 \text{ at } r = R(0,\theta), \quad G(\boldsymbol{x};\boldsymbol{\xi}) = 0 \text{ at } x = 0,$$

$$G(\boldsymbol{x};\boldsymbol{\xi}) \to \text{a constant for } x \to \infty, \qquad x\frac{\partial}{\partial x}G(\boldsymbol{x};\boldsymbol{\xi}) \to 0 \text{ for } x \to \infty.$$

We determine the Green's function by applying the Fourier sine transform[3] with respect to x ($x \to \alpha$) to (43), to obtain

$$(44) \qquad \nabla_\perp^2\hat{G} - \alpha^2\hat{G} = -\sqrt{\frac{2}{\pi}}\sin(\alpha\xi)\delta(\boldsymbol{x}_\perp - \boldsymbol{\xi}_\perp),$$

where $\boldsymbol{x}_\perp$ denotes the transverse component of $\boldsymbol{x}$, i.e., $\boldsymbol{x}_\perp = (r,\theta)$ (similarly for $\boldsymbol{\xi}_\perp$). We assume that the Green's function can be expanded by the same basis function as has been used for Φ_0,

$$\hat{G}(\alpha,r,\theta;\boldsymbol{\xi}) = \sum_{m=0}^\infty a_m(\alpha,\boldsymbol{\xi})\psi_m(r,\theta).$$

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[3]Here $\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\alpha x)f(x)\,\mathrm{d}x$, $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\alpha x)\hat{f}(\alpha)\,\mathrm{d}\alpha$.

Therefore

$$\nabla^2 \hat{G} = -\sum_{m=0}^{\infty} a_m \lambda_m^2 \psi_m(r, \theta).$$

Substituting this into (44) yields

$$(45) \qquad \sum_{m=0}^{\infty} a_m \psi_m (\lambda_m^2 + \alpha^2) = \sqrt{\frac{2}{\pi}} \sin(\alpha\xi) \delta(\boldsymbol{x}_\perp - \boldsymbol{\xi}_\perp).$$

Next, we multiply (45) with ψ_n and integrate over the cross section $\mathcal{A}(0)$ to obtain

$$(46) \qquad \iint_{\mathcal{A}(0)} \sum_{m=0}^{\infty} a_m \psi_n \psi_m (\lambda_m^2 + \alpha^2) \, d\sigma = \sqrt{\frac{2}{\pi}} \iint_{\mathcal{A}(0)} \psi_n(r, \theta) \sin(\alpha\xi) \delta(\boldsymbol{x}_\perp - \boldsymbol{\xi}_\perp) \, d\sigma.$$

Orthonormality of the basis functions yields

$$(47) \qquad a_m = \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\alpha\xi)}{\lambda_m^2 + \alpha^2} \right) \psi_m(\rho, \eta).$$

Therefore,

$$(48) \qquad \hat{G}(\alpha, r, \theta; \xi, \rho, \eta) = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \frac{\sin(\alpha\xi)}{\lambda_m^2 + \alpha^2} \psi_m(\rho, \eta) \psi_m(r, \theta).$$

The inverse Fourier sine transform yields

$$(49) \qquad G(\boldsymbol{x}; \boldsymbol{\xi}) = \frac{2}{\pi} \sum_{m=0}^{\infty} \psi_m(\rho, \eta) \psi_m(r, \theta) \int_0^\infty \frac{\sin(\alpha x) \sin(\alpha\xi)}{\lambda_m^2 + \alpha^2} \, d\alpha,$$

where [25] for $\lambda_0 = 0$,

$$(50) \qquad \int_0^\infty \frac{\sin(\alpha x) \sin(\alpha\xi)}{\alpha^2} \, d\alpha = \frac{1}{2} \pi \min(x, \xi),$$

and for $\lambda_m > 0$,

$$(51) \qquad \int_0^\infty \frac{\sin(\alpha x) \sin(\alpha\xi)}{\lambda_m^2 + \alpha^2} \, d\alpha = \frac{1}{2} \pi \, e^{-\lambda_m \max(x,\xi)} \frac{1}{\lambda_m} \sinh(\lambda_m \min(x, \xi)).$$

Therefore, the $m = 0$ term can be taken apart, and the Green's function becomes

$$(52a) \qquad G(\boldsymbol{x}; \boldsymbol{\xi}) = x + \sum_{m=1}^{\infty} \psi_m(\rho, \eta) \psi_m(r, \theta) \, e^{-\lambda_m \xi} \frac{\sinh(\lambda_m x)}{\lambda_m} \qquad \text{if } 0 \le x \le \xi$$

$$(52b) \qquad = \xi + \sum_{m=1}^{\infty} \psi_m(\rho, \eta) \psi_m(r, \theta) \, e^{-\lambda_m x} \frac{\sinh(\lambda_m \xi)}{\lambda_m} \qquad \text{if } 0 \le \xi \le x.$$

Note that as $x \to \infty$, G tends to ξ and $\frac{\partial G}{\partial x}$ tends to zero exponentially.

Using this Green's function, we obtain for Φ_1 the following relation, to be integrated over domain $\mathcal{V}$:

$$(53) \qquad \Phi_1 \delta(\boldsymbol{x} - \boldsymbol{\xi}) = G \nabla^2 \Phi_1 - \Phi_1 \nabla^2 G.$$

However, since $\Phi_1 \sim K_1\xi$ for large ξ (see the remark below (32)), this yields a divergent integral as the domain here is a semi-infinite duct. Therefore, we consider a region $\mathcal{V}'$ with a finite length $0 \leq x \leq x_0$, where x_0 is small compared to ε^{-1} but large enough for all exponential terms to practically vanish. Integrate (53) along domain $\mathcal{V}'$ and by using Green's second identity we get

$$\Phi_1(\boldsymbol{\xi}) = \iiint_{\mathcal{V}'} \left(G\nabla^2\Phi_1 - \Phi_1\nabla^2 G\right) \mathrm{d}\boldsymbol{x} = \iint_{x=0} \left(-G\frac{\partial\Phi_1}{\partial x} + \Phi_1\frac{\partial G}{\partial x}\right) \mathrm{d}\sigma$$

$$+ \iint_{r=R(0,\eta)} \left(G\nabla_\perp\Phi_1 - \Phi_1\nabla_\perp G\right)\cdot\boldsymbol{n}_\perp \, \mathrm{d}\sigma + \iint_{x=x_0} \left(G\frac{\partial\Phi_1}{\partial x} - \Phi_1\frac{\partial G}{\partial x}\right) \mathrm{d}\sigma$$

$$(54) \qquad = \iint_{r=R(0,\eta)} \frac{G\mathcal{Q}_0(x,\theta)}{|\nabla_\perp\Sigma|} \, \mathrm{d}\ell\mathrm{d}\xi + K_1\xi.$$

Since $|\nabla_\perp\Sigma| = \frac{1}{R}\sqrt{R^2 + R_\theta^2}$ and $\mathrm{d}\ell = \sqrt{R^2 + R_\theta^2}\,\mathrm{d}\theta$, we obtain

$$(55) \qquad \Phi_1(\boldsymbol{\xi}) = \int_0^{2\pi}\int_0^\infty \mathcal{Q}_0(x,\theta)G(\boldsymbol{x};\boldsymbol{\xi})|_{r=R}R\,\mathrm{d}x\mathrm{d}\theta + K_1\xi.$$

As we have $\mathcal{Q}_0$ in the form of a series expansion, we can write

$$(56) \quad \Phi_1(\boldsymbol{\xi}) = K_1\xi + \sum_{n=1}^\infty F_n \int_0^{2\pi}\left[-RR_X\lambda_n\psi_n\big|_{r=R}\int_0^\infty \mathrm{e}^{-\lambda_n x}\,G(\boldsymbol{x};\boldsymbol{\xi})\big|_{r=R}\,\mathrm{d}x\right.$$

$$\left. + \left\{RR_X\lambda_n^2\psi_n\big|_{r=R} + \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_X}{R}\psi_{n,\theta}\big|_{r=R}\right)\right\}\int_0^\infty x\,\mathrm{e}^{-\lambda_n x}\,G(\boldsymbol{x};\boldsymbol{\xi})\big|_{r=R}\,\mathrm{d}x\right]\mathrm{d}\theta.$$

As the series for $\mathcal{Q}_0$ converges uniformly for $x > 0$, we may exchange summation and integration. On the other hand, the fact that all basis functions have vanishing normal derivatives at the wall, i.e., $\nabla_\perp\psi_n\cdot\boldsymbol{n}_\perp = 0$, whereas $\nabla_\perp\Phi_1\cdot\boldsymbol{n}_\perp \neq 0$, suggests that this series does not converge uniformly near the wall.

The expression for Φ_1 is further specified by removing the x-integration:

$$(57)$$
$$\int_0^\infty \mathrm{e}^{-\lambda_n x}\,G(\boldsymbol{x};\boldsymbol{\xi})\Big|_{r=R}\,\mathrm{d}x = \frac{1 - \mathrm{e}^{-\lambda_n\xi}}{\lambda_n^2} - \sum_{m=1}^\infty \psi_m(R,\theta)\psi_m(\rho,\eta)\frac{\mathrm{e}^{-\lambda_n\xi} - \mathrm{e}^{-\lambda_m\xi}}{\lambda_n^2 - \lambda_m^2},$$

$$(58) \quad \int_0^\infty x\,\mathrm{e}^{-\lambda_n x}\,G(\boldsymbol{x};\boldsymbol{\xi})\Big|_{r=R}\,\mathrm{d}x = \frac{2 - (2 + \lambda_n\xi)\,\mathrm{e}^{-\lambda_n\xi}}{\lambda_n^3}$$

$$- \sum_{m=1}^\infty \psi_m(R,\theta)\psi_m(\rho,\eta)\frac{2\lambda_n(\mathrm{e}^{-\lambda_n\xi} - \mathrm{e}^{-\lambda_m\xi}) + \xi(\lambda_n^2 - \lambda_m^2)\,\mathrm{e}^{-\lambda_n\xi}}{(\lambda_n^2 - \lambda_m^2)^2}.$$

If $m = n$, the limit $\lambda_m \to \lambda_n$ should be taken. Now we are better able to recognize the nature of the nonuniform convergence. The dominating term is (we ignore for the moment the θ-integration)

$$\Phi_1(\boldsymbol{\xi}) \sim \sum_{m=1}^\infty \frac{\psi_m(R,\theta)\psi_m(\rho,\eta)}{\lambda_m^2}.$$

For a circular duct this may be compared, near $\rho = R$, to the prototype series

$$\sim \sum_{m=1}^{\infty} \frac{\cos(2\pi m\rho/R)}{m^2}.$$

The normal derivative yields the well-known saw-tooth function that vanishes (pointwise) at $\rho = R$ but converges to a finite nonzero value for any $\rho \neq R$.

For $x \to \infty$, the exponential terms in $\Phi_1(\boldsymbol{x})$ vanish and we have (we exchange the variables $\boldsymbol{x}$ and $\boldsymbol{\xi}$)

$$\Phi_1(\boldsymbol{x}) \simeq K_1 x + \sum_{n=1}^{\infty} F_n \int_0^{2\pi} \left[RR_X \lambda_n^{-1} \psi_n \big|_{\rho=R} + \frac{2}{\lambda_n} \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\frac{R_X}{R} \psi_{n,\eta} \big|_{\rho=R} \right) \right] \mathrm{d}\eta.$$

By using the periodicity of ψ_n in its circumferential argument η, we have finally

$$(59) \qquad \Phi_1(\boldsymbol{x}) \simeq K_1 x + \sum_{n=1}^{\infty} \frac{F_n}{\lambda_n} \int_0^{2\pi} RR_X \psi_n \big|_{\rho=R} \mathrm{d}\eta \qquad \text{for} \ \ x \to \infty.$$

4.3. Matching. Both the initial conditions for ϕ_0 and ϕ_1 and the constant K_1 are determined from matching with the outer solution. From (41) and (59) we have

$$(60) \qquad \phi_0(0) + X\phi_{0,X}(0) + \varepsilon\phi_1(0) \sim F_0 + \varepsilon K_1 x + \varepsilon \sum_{n=1}^{\infty} \frac{F_n}{\lambda_n} \int_0^{2\pi} RR_X \psi_n \big|_{\rho=R} \mathrm{d}\eta,$$

and so we find

$$(61) \qquad \begin{cases} \phi_0(0) = F_0, \\ \quad K_1 = \phi_{0,X}(0), \\ \phi_1(0) = \displaystyle\sum_{n=1}^{\infty} \frac{F_n}{\lambda_n} \int_0^{2\pi} RR_X \psi_n \big|_{\rho=R} \mathrm{d}\eta. \end{cases}$$

This determines the outer solution $\phi_0 + \varepsilon\phi_1$ (together with the radiation condition). It wouldn't be too difficult to guess that ϕ_0 depends on the average source excitation F_0, but the initial value for ϕ_1 is really subtle. The constant term in (59) is therefore probably the most important result of this tour de force to determine Φ_1.

An interesting question is then when ϕ_1 is present at all in the outer solution (or put in another way: what the error is if we only consider ϕ_0). For example, ϕ_1 is zero when the source consists of a simple piston with just $F(r, \theta) = F_0$, or when the duct entrance starts smoothly with $R_X = 0$, or when $RR_X \psi_n$ for all $n > 0$ are periodic along the circumference.

Although this last condition is not very likely to be possible, for a cylindrical duct at least the nonsymmetric modes vanish. In this case the eigenfunctions are given by (38). The integrals in (59) vanish for all $\nu \neq 0$. As a result we have

$$(62) \qquad \phi_1(0) = 2\sqrt{\pi} RR_X \sum_{\mu=2}^{\infty} \frac{F_{0\mu}}{j'_{0\mu}}.$$

In other words, the first constant mode determines ϕ_0, while only the nonconstant symmetric modes determine ϕ_1. For example, a piston tilting along a diagonal like $F \sim r\sin\theta$ would produce a field vanishing to $\mathcal{O}(\varepsilon^2)$, while a "piston" that is symmetrically folded like $F \sim r^2$ would produce both $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ terms.

5. Other coordinate systems. It was shown by Agullo, Barjau, and Keefe [26] that if the shape of the hard-walled duct is described in an orthogonal coordinate system (u, v, w) by the surface $\Sigma(v, w) = 0$, while the Helmholtz equation allows separable solutions of the form $\phi(u, v, w) = F(u)G(v, w)$, then there exist unidimensional (i.e., self-similar) waves in u of the type $\phi(u, v, w) = F(u)$. In this way it is possible to produce exact solutions of certain horn shapes, like the straight and exponential cone and others.

Although these solutions are interesting on their own, they have little to do with the present low k asymptotic problem, where the duct wall is never outside the lateral near field of the wave. Without this, there is no built-in mechanism that enforces the self-similarity, so any defect of symmetry in source or surface will produce deviations in the wave field that propagate without attenuation in other directions. Also the generalizations that will be discussed below are not possible at all or only in very limited form.

On the other hand, if the duct shape considered is close to one that allows such an exact solution, it may be advantageous, in terms of practical accuracy of the final result, to reformulate the problem in the other set of coordinates. The essence of the asymptotic problem remains the same.

We will illustrate this for spherical coordinates (r, θ, φ), where we temporarily redefine $x = r \cos \varphi$, $y = r \sin \varphi \cos \theta$, $z = r \sin \varphi \sin \theta$. (Note that we will use these coordinates *only in this section*.) A circular cone around the positive x-axis is given by $\varphi = $ constant, and a general cone of constant cross section by $\varphi = f(\theta)$.

In order to maintain the slender shape, necessary for the asymptotics, the duct will be long in r, compensated by a small opening angle in φ. We therefore introduce the scaled variables

$$(63) \qquad\qquad \tau = \frac{2 \sin \frac{1}{2}\varphi}{\varepsilon}, \qquad R = \varepsilon r$$

and write the general duct geometry as

$$(64) \qquad\qquad \tilde{\Sigma}(R, \tau, \theta) = \tau - T(R, \theta) = 0,$$

where T is, by assumption, independent of ε. By this choice the surface area, $\tilde{A}(R)$ of any spherical cross section $R = $ constant is now exactly (i.e., independent of ε) equal to

$$\tilde{A}(R) = \int_0^{2\pi} \int_0^{\varphi(R,\theta)} r^2 \sin \varphi \, \mathrm{d}\varphi \mathrm{d}\theta = \int_0^{2\pi} \int_0^T r^2 \varepsilon^2 \tau \, \mathrm{d}\tau \mathrm{d}\theta$$

$$(65) \qquad\qquad = \frac{1}{2} R^2 \int_0^{2\pi} T^2(R, \theta) \, \mathrm{d}\theta.$$

Other choices for describing the duct shape are not essentially different, other than T, and therefore $\tilde{A}$, becoming dependent on ε. This gives complications in the form of extra asymptotic terms in the higher orders, which are irrelevant now.

The Helmholtz equation is given by

$$(66)$$
$$\frac{\varepsilon^2}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right) + \frac{1}{R^2 \tau} \frac{\partial}{\partial \tau}\left(\tau \left(1 - \frac{1}{4}\varepsilon^2 \tau^2\right) \frac{\partial \phi}{\partial \tau}\right) + \frac{1}{R^2 \tau^2 (1 - \frac{1}{4}\varepsilon^2 \tau^2)} \frac{\partial^2 \phi}{\partial \theta^2} + \varepsilon^2 \kappa^2 \phi = 0,$$

while the hard-wall boundary condition becomes

$$(67) \qquad \nabla\phi\cdot\nabla\tilde{\Sigma} = \frac{1-\frac{1}{4}\varepsilon^2 T^2}{R^2}\frac{\partial\phi}{\partial\tau} - \varepsilon^2\frac{\partial T}{\partial R}\frac{\partial\phi}{\partial R} - \frac{1}{R^2 T^2(1-\frac{1}{4}\varepsilon^2 T^2)}\frac{\partial T}{\partial\theta}\frac{\partial\phi}{\partial\theta} = 0.$$

We expand, as before,

$$\phi(R,\tau,\theta;\varepsilon) = \phi_0(R,\tau,\theta) + \varepsilon^2\phi_2(R,\tau,\theta) + \cdots$$

(skipping for now the $\mathcal{O}(\varepsilon)$-term) to obtain to leading order

$$(68) \qquad \phi_{0,\tau\tau} + \frac{1}{\tau}\phi_{0,\tau} + \frac{1}{\tau^2}\phi_{0,\theta\theta} = 0, \quad \text{with} \quad \phi_{0,\tau} - \frac{T_\theta}{T^2}\phi_{0,\theta} = 0 \quad \text{at} \quad \tau = T.$$

If τ and θ are read as polar coordinates, this problem is qua form the same as (17), and thus we have the solution $\phi_0 = \phi_0(R)$ to be determined at the next order. We have

$$\phi_{2,\tau\tau} + \frac{1}{\tau}\phi_{2,\tau} + \frac{1}{\tau^2}\phi_{2,\theta\theta} + \left(R^2\phi_{0,R}\right)_R + R^2\kappa^2\phi_0 = 0,$$

$$\text{with} \quad \phi_{2,\tau} - \frac{T_\theta}{T^2}\phi_{2,\theta} = R^2 T_R\phi_{0,R} \quad \text{at} \quad \tau = T.$$

This can be written as

$$(69) \qquad \tilde{\nabla}^2\phi_2 + \left(R^2\phi_{0,R}\right)_R + R^2\kappa^2\phi_0 = 0, \quad \text{with} \quad \tilde{\nabla}\phi_2\cdot\tilde{\boldsymbol{n}} = R^2\phi_{0,R}\frac{TT_R}{\sqrt{T^2+T_\theta^2}},$$

where $\tilde{\nabla}$ and $\tilde{\boldsymbol{n}}$ denote gradient and normal, respectively, in the (τ,θ)-plane. As a result we have virtually the same equation as (19), and after integration along a spherical surface $\tilde{A}(R)$ in (τ,θ) and using (65), we obtain

$$-\frac{\tilde{A}}{R^2}\left(R^2\phi_{0,R}\right)_R - \tilde{A}\kappa^2\phi_0 = \frac{1}{2}R^2\phi_{0,R}\frac{\mathrm{d}}{\mathrm{d}R}\int_0^{2\pi} T^2(R,\theta)\,\mathrm{d}\theta = R^2\phi_{0,R}\left(\frac{\tilde{A}}{R^2}\right)_R$$

or

$$(70) \qquad \tilde{A}^{-1}\left(\tilde{A}\phi_{0,R}\right)_R + \kappa^2\phi_0 = 0.$$

We see that changing from the axial coordinate X to R and from the transverse cross section A to the spherical cross section $\tilde{A}$ leaves the final equation for ϕ_0 unchanged. Indeed, to the order considered, X and R and A and $\tilde{A}$ are the same.

6. Curved ducts. The present results remain valid for the slightly more general problem of curved ducts (like certain musical instruments) if the curvature of the duct axis (and its derivative) is $\mathcal{O}(\varepsilon)$. Together with the assumed slow variation in the axial coordinate, the associated orthogonal coordinate system (based on the tangent and, possibly, the normal and binormal of the curve that describes the duct axis) leave the Laplacian unchanged up to $\mathcal{O}(\varepsilon^3)$.

A simple example is the inside of a perturbed torus, described by a fixed torus radius ε^{-1} and slowly varying tube radius R. With local (polar-type) coordinates ξ, r, φ, we define

$$(71) \qquad x = \varepsilon^{-1}(1+\varepsilon r\cos\theta)\cos(\varepsilon\xi), \quad y = \varepsilon^{-1}(1+\varepsilon r\cos\theta)\sin(\varepsilon\xi), \quad z = r\sin\theta,$$
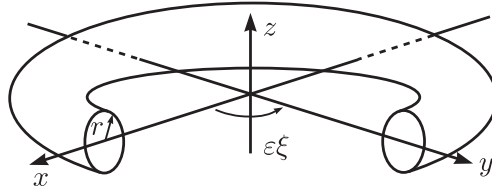
FIG. 3. *The torus coordinates.*

where $0 \leq r \leq R(\varepsilon\xi, \theta)$, $0 \leq \theta < 2\pi$, $0 \leq \varepsilon\xi < 2\pi$ (see Figure 3). If we write $X = \varepsilon\xi$, we get (cf. (6))

$$(72) \quad \nabla^2 \phi + \varepsilon^2 \kappa^2 \phi$$

$$= \nabla_\perp^2 \phi + \varepsilon^2 (1 + \varepsilon r \cos\theta)^{-2} \frac{\partial^2}{\partial X^2} \phi + \varepsilon (1 + \varepsilon r \cos\theta)^{-1} \left[\cos\theta \frac{\partial}{\partial r} \phi - \frac{1}{r} \frac{\partial}{\partial \theta} \phi \right] + \varepsilon^2 \kappa^2 \phi = 0.$$

Boundary conditions at $\Sigma = r - R(X, \theta) = 0$ are

$$(73) \qquad\qquad \nabla_\perp \phi \cdot \nabla_\perp \Sigma - \frac{\varepsilon^2 R_X \phi_X}{(1 + \varepsilon r \cos\theta)^2} = 0.$$

If we expand $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$, we get to leading order

$$(74) \qquad\qquad \nabla_\perp^2 \phi_0 = 0, \qquad \text{with} \quad \nabla_\perp \phi_0 \cdot \boldsymbol{n}_\perp = 0,$$

and so $\phi_0 = \phi_0(X)$. Then $\frac{\partial}{\partial r}\phi_0 = \frac{\partial}{\partial\theta}\phi_0 = 0$, and we also have

$$(75) \qquad\qquad \nabla_\perp^2 \phi_1 = 0, \qquad \text{with} \quad \nabla_\perp \phi_1 \cdot \boldsymbol{n}_\perp = 0,$$

leading to $\phi_1 = \phi_1(X)$. Thus again $\frac{\partial}{\partial r}\phi_1 = \frac{\partial}{\partial\theta}\phi_1 = 0$, and we again obtain

$$\nabla_\perp^2 \phi_2 + \phi_{0,XX} + \kappa^2 \phi_0 = 0, \qquad \text{with} \quad \nabla_\perp \phi_2 \cdot \nabla_\perp \Sigma = \phi_{0,X} R_X,$$

yielding thus, after a similar argument as before, Webster's horn equation.

7. Impedance walls. If the duct walls is equipped with an impedance-type acoustic lining of complex impedance Z, we will in general (at least if $\text{Re}(Z) > 0$) expect solutions that decay exponentially in the axial direction. Therefore, in the compressed variable X, only trivial (i.e., zero) solutions will exist. We will see that this is by and large the case, not only for dissipative walls with $\text{Re}(Z) > 0$, but for any $|Z| < \infty$. Only for a purely imaginary impedance in a *straight* duct are there exceptions.

The impedance-wall boundary condition at $r = R$ is given by

$$(76) \qquad\qquad \nabla\phi \cdot \boldsymbol{n} = -\frac{\mathrm{i}\varepsilon\kappa}{Z}\phi = \zeta\phi$$

with specific impedance Z. As before, we assume the Poincaré expansion $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$. First we note that it is easily verified that if $Z = 0$, only the trivial solutions $\phi_0 = \phi_1 = 0$ occur. Then we consider two possibilities: $Z = \mathcal{O}(1)$ and $Z = \mathcal{O}(\varepsilon)$.

7.1. $Z = \mathcal{O}(1)$. As $\zeta = \mathcal{O}(\varepsilon)$, we write $\zeta = \varepsilon\zeta_1$. In this case we have only trivial solutions. Expand equations and boundary conditions as before, to get to leading order

$$(77) \qquad \nabla_\perp^2 \phi_0 = 0, \quad \text{with} \quad \nabla_\perp \phi_0 \cdot \boldsymbol{n}_\perp = 0,$$

with solution $\phi_0 = \phi_0(X)$, a function to be determined. To first order we have

$$(78) \qquad \nabla_\perp^2 \phi_1 = 0, \quad \text{with} \quad \nabla_\perp \phi_1 \cdot \boldsymbol{n}_\perp = \zeta_1 \phi_0.$$

Since

$$(79) \qquad \iint_{\mathcal{A}} \nabla_\perp^2 \phi_1 \, d\sigma = \zeta_1 \phi_0 \int_{\partial\mathcal{A}} d\ell = 0,$$

we must have $\phi_0 = 0$, and so $\phi_1 = \phi_1(X)$. Nothing changes when we continue, and so all terms of the expansion vanish. Note that this is true for any Z.

7.2. $Z = \mathcal{O}(\varepsilon)$. Now we have $\zeta = \mathcal{O}(1)$, which changes the boundary condition expansion. To leading order we have

$$(80) \qquad \nabla_\perp^2 \phi_0 = 0 \quad \text{in} \quad \mathcal{A}, \quad \text{with} \quad \nabla\phi_0 \cdot \boldsymbol{n}_\perp = \zeta\phi_0 \quad \text{at} \quad \partial\mathcal{A}.$$

We would be tempted to assume that this problem has a solution or solutions for any given Z, but this is not true. Nontrivial solutions exist only for certain ζ. From Green's second identity applied to ϕ_0 and its complex conjugate, it can be deduced that any possible ζ is real. Furthermore, from Green's first identity applied to ϕ_0, it follows that any possible ζ is positive, and Z is thus negative imaginary.

But even with ζ real positive, there are only certain discrete values that allow a solution. This is best seen as follows. The problem described in (80) is an eigenvalue problem for the Dirichlet-to-Neumann operator $\Xi: f \mapsto g$, which maps a given Dirichlet boundary value f to the normal derivative g of f's harmonic extension into $\mathcal{A}$ (see [17]). In other words, $\Xi(f) = \frac{\partial}{\partial n}\psi\big|_{\partial\mathcal{A}}$, where ψ is the solution of

$$(81) \qquad \nabla^2\psi = 0 \quad \text{in} \quad \mathcal{A}, \quad \text{with} \quad \psi = f \quad \text{at} \quad \partial\mathcal{A}.$$

As we are looking for $\Xi(\phi_0) = \zeta\phi_0$, equation (80) corresponds to the eigenvalue problem of Ξ. For the present discussion it is most relevant to know that this spectrum of eigenvalues of Ξ is discrete. As this result, due to Prof. Jan de Graaf, appears not to be available in the literature, it is considered concisely, but in great depth, in the appendix.

An example that illustrates this behavior explicitly is the circular duct $r = R(X)$, where

$$(82) \qquad \phi_0 = f(X)\Big(\frac{r}{R(X)}\Big)^m \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}, \quad \text{with} \quad \zeta = \frac{m}{R(X)},$$

and m is a nonnegative integer. As the shape of the cross section $\mathcal{A}(X)$ changes with X, the discreteness of the spectrum of Ξ implies that the values of ζ that allow a solution also change with X, and in general there are no (nonzero) solutions possible along a varying duct for a fixed given ζ.

This is of course not true for a duct of constant cross section, $r = R(\theta)$, although now the asymptotics for small ε loses its meaning because there is no axial length

scale for the acoustic wave to be compared with. The problem simplifies further for the circular duct $r = R$, where (without approximation)

$$(83) \qquad \phi(x, r, \theta) = J_m(\alpha r)\, e^{-im\theta - i\gamma x}, \quad \alpha^2 + \gamma^2 = k^2$$

and the boundary condition requires

$$(84) \qquad \frac{\alpha R J_m'(\alpha R)}{J_m(\alpha R)} = m - \frac{\alpha R J_{m+1}(\alpha R)}{J_m(\alpha R)} = \zeta R.$$

This equation has infinitely many solutions, but the wave is guaranteed unattenuated (γ real) if α is imaginary, say $\alpha = i\tau$. Such solutions exist for real $\zeta \geq m/R$, because

$$(85) \qquad \zeta R = m - \frac{i\tau R J_{m+1}(i\tau R)}{J_m(i\tau R)} = m + \frac{\tau R I_{m+1}(\tau R)}{I_m(\tau R)} \geq m$$

(see [27]). Note that for small k, γ, α solutions we recover (82)

$$(86) \qquad \zeta = \frac{m}{R} - \frac{\alpha^2 R}{2m + 2} + \mathcal{O}(\alpha^4).$$

In other words, only solutions of this type exist near special values of ζ.

8. Variable mean soundspeed and density. If soundspeed $C = C(X, r, \theta)$ and mean density $D = D(X, r, \theta)$ are not uniformly constant, but vary in r, θ, and slowly in x, we have the reduced wave equation (5), rewritten in slowly varying coordinates as

$$(87) \qquad \varepsilon^2 \frac{\partial}{\partial X}\left(C^2 p_X\right) + \nabla_\perp \cdot \left(C^2 \nabla_\perp p\right) + \varepsilon^2 \Omega^2 p = 0,$$

where the dimensionless frequency $\omega = \varepsilon\Omega$ is small. The hard-wall boundary condition is the same as (14). When we expand $p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots$, we get to leading order

$$(88) \qquad \nabla_\perp \cdot \left(C^2 \nabla_\perp p_0\right) = 0, \quad \text{with} \quad \nabla_\perp p_0 \cdot \boldsymbol{n}_\perp = 0,$$

which has a constant as the solution, so $p_0 = p_0(X)$, a function to be determined. We can derive the same equation for p_1, to get the same result $p_1 = p_1(X)$. For the second order we have

$$(89)$$
$$\nabla_\perp \cdot \left(C^2 \nabla_\perp p_2\right) + \frac{\partial}{\partial X}\left(C^2 p_{0,X}\right) + \Omega^2 p_0 = 0, \quad \text{with} \quad \nabla_\perp p_2 \cdot \boldsymbol{n}_\perp = p_{0,X} \frac{RR_X}{\sqrt{R^2 + R_\theta^2}}.$$

We go on to find a solvability condition for p_2 by integrating this equation along a cross section $\mathcal{A}$. Utilizing the following identity for any differentiable function f,

$$
\frac{\mathrm{d}}{\mathrm{d}X} \iint_{\mathcal{A}} f(\boldsymbol{X})\, \mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}X} \int_0^{2\pi} \int_0^R f(X, r, \theta) r\, \mathrm{d}r \mathrm{d}\theta
$$
$$(90) \qquad\qquad = \int_0^{2\pi} \int_0^R f_X r\, \mathrm{d}r\mathrm{d}\theta + \int_0^{2\pi} f(X, R, \theta) RR_X\, \mathrm{d}\theta,$$

we have

$$\iint_{\mathcal{A}} \nabla_\perp \cdot (C^2 \nabla_\perp p_2) \, d\sigma = p_{0,X} \int_0^{2\pi} C^2 R R_X \, d\theta$$

(91a)
$$= p_{0,X} \left[\frac{d}{dX} \iint_{\mathcal{A}} C^2 \, d\sigma - \iint_{\mathcal{A}} \frac{\partial}{\partial X} C^2 \, d\sigma \right].$$

Furthermore, we have

$$\iint_{\mathcal{A}} \frac{\partial}{\partial X} (C^2 p_{0,X}) \, d\sigma = p_{0,X} \iint_{\mathcal{A}} \frac{\partial}{\partial X} C^2 \, d\sigma + p_{0,XX} \iint_{\mathcal{A}} C^2 \, d\sigma,$$

(91b)   and $$\iint_{\mathcal{A}} \Omega^2 p_0 \, d\sigma = \Omega^2 p_0 A.$$

Then, after introducing the cross-sectional averaged squared soundspeed

(92)
$$\overline{C^2} = \frac{1}{A} \iint_{\mathcal{A}} C^2 \, d\sigma,$$

a generalization of Webster's horn equation is obtained:

(93)
$$A^{-1} \left(A \overline{C^2} p_{0,X} \right)_X + \Omega^2 p_0 = 0.$$

This may be further simplified by the transformation

(94)
$$A(X) \overline{C^2}(X) = d(X)^2, \quad p_0 = d^{-1} \psi$$

into

(95)
$$\psi'' + \left(\frac{\Omega^2}{\overline{C^2}} - \frac{d''}{d} \right) \psi = 0.$$

9. Irrotational and isentropic mean flow. To analyze asymptotically low-frequency acoustic perturbations in a slowly varying duct with an irrotational isentropic mean flow, as described by (7) and (9), we need to approximate both mean flow and acoustic field to the same order of accuracy.

We start here with the mean flow. In the dimensionless variables used, we have $C^2 = D^{\gamma-1}$, so equations (7) simplify to

(96)
$$\tfrac{1}{2} V^2 + \frac{D^{\gamma-1}}{\gamma - 1} = E, \qquad \nabla \cdot (D\boldsymbol{V}) = 0.$$

The mass flux at any cross section A is given by

(97)
$$\iint_{\mathcal{A}} DU \, d\sigma = \mathcal{F}.$$

Due to the nondimensionalization, U, D, A, $\mathcal{F}$, and E are $\mathcal{O}(1)$. Introduce the slow variable $X = \varepsilon x$, and assume that $\boldsymbol{V}$ and D depend essentially on X, rather than x. We write the velocity as

(98)
$$\boldsymbol{V} = U\boldsymbol{e}_x + \boldsymbol{V}_\perp$$

to distinguish between axial and crosswise components. If flux $\mathcal{F}$ and thermodynamical constant E are given and independent of ε, we can expand $U = U_0 + \mathcal{O}(\varepsilon^2)$ and $D = D_0 + \mathcal{O}(\varepsilon^2)$. As the flow is a potential flow, we can derive, in the same way as in Rienstra [14, 16], that $D_0 = D_0(X)$, $U_0 = U_0(X)$, and $\boldsymbol{V}_\perp = \varepsilon\widetilde{\boldsymbol{V}}_{\perp 0} + \mathcal{O}(\varepsilon^3)$, satisfying the equations (to be solved numerically)

$$(99) \qquad D_0 U_0 A = \mathcal{F}, \qquad \frac{\mathcal{F}^2}{2D_0^2 A^2} + \frac{D_0^{\gamma-1}}{\gamma-1} = E.$$

9.1. Mean flow and hard walls. Next we consider the acoustic field. Using the above results for the mean flow, (9) becomes to leading order

$$\nabla_\perp^2 \phi + \varepsilon^2 D_0^{-1}\left(D_0 \phi_X\right)_X$$
$$= \varepsilon^2 \left(\mathrm{i}\Omega + U_0 \frac{\partial}{\partial X} + \widetilde{\boldsymbol{V}}_{\perp 0}{\boldsymbol{\cdot}}\nabla_\perp\right)\left[C_0^{-2}\left(\mathrm{i}\Omega + U_0\frac{\partial}{\partial X} + \widetilde{\boldsymbol{V}}_{\perp 0}{\boldsymbol{\cdot}}\nabla_\perp\right)\phi\right],$$

with hard wall boundary condition

$$\nabla\phi{\boldsymbol{\cdot}}\boldsymbol{n} = 0 \quad \text{at} \quad r = R.$$

We expand $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$. To leading order we have

$$(100) \qquad \nabla_\perp^2 \phi_0 = 0, \quad \nabla_\perp\phi_0{\boldsymbol{\cdot}}\boldsymbol{n}_\perp = 0,$$

yielding the constant solution, i.e., $\phi_0 = \phi_0(X)$.

To first order we have the same equation. To second order we have

$$\nabla_\perp^2 \phi_2 + D_0^{-1}\left(D_0\phi_{0,X}\right)_X$$
$$= \left(\mathrm{i}\Omega + U_0 \frac{\partial}{\partial X} + \widetilde{\boldsymbol{V}}_{\perp 0}{\boldsymbol{\cdot}}\nabla_\perp\right)\left[C_0^{-2}\left(\mathrm{i}\Omega + U_0\frac{\partial}{\partial X} + \widetilde{\boldsymbol{V}}_{\perp 0}{\boldsymbol{\cdot}}\nabla_\perp\right)\phi_0\right],$$

with boundary conditions given by (19). After integration across a cross section $\mathcal{A}(X)$, we obtain, similar to before, Webster's horn equation generalized for irrotational isentropic mean flow:

$$(101) \qquad (D_0 A)^{-1}(D_0 A\phi_{0,X})_X = \left(\mathrm{i}\Omega + U_0 \frac{\partial}{\partial X}\right)\left[C_0^{-2}\left(\mathrm{i}\Omega + U_0\frac{\partial}{\partial X}\right)\phi_0\right].$$

This result seems to be equivalent to equations given by [8, 9, 10, 11, 12] and (apart from a factor $\frac{1}{2}$) [3, p. 422].

9.2. Mean flow and impedance walls. The problem with mean flow and an impedance wall is more intricate. Instead of the duct wall boundary condition given in (76), we have Myers' condition [28], rewritten (see [29, 30]) as follows:

$$(102) \qquad \mathrm{i}\omega D\left(\boldsymbol{v}{\boldsymbol{\cdot}}\boldsymbol{n}\right) = \frac{\mathrm{i}\omega Dp}{Z} + \mathcal{M}\left(\frac{D\boldsymbol{V}p}{Z}\right),$$

where impedance $Z = Z(X, \theta)$ may be function of position, and operator $\mathcal{M}$ is defined by

$$(103) \qquad \mathcal{M}(\boldsymbol{F}) = \nabla{\boldsymbol{\cdot}}\boldsymbol{F} - \boldsymbol{n}{\boldsymbol{\cdot}}(\boldsymbol{n}{\boldsymbol{\cdot}}\nabla\boldsymbol{F}).$$

Since $\mathcal{M}(\frac{D\boldsymbol{V}p}{Z}) = \mathcal{O}(\varepsilon)$, we write $\mathcal{M}(\frac{D\boldsymbol{V}p}{Z}) = \varepsilon\widetilde{\mathcal{M}}(\frac{D\boldsymbol{V}p}{Z})$. After expanding $\phi = \phi_0 + \varepsilon\phi_1 + \cdots$ and $p = \varepsilon p_0 + \cdots$ with

(104)
$$p_0 = -D_0\left(i\Omega + U_0\frac{\partial}{\partial X} + \widetilde{\boldsymbol{V}}_{\perp 0}\boldsymbol{\cdot}\nabla_\perp\right)\phi_0,$$

we get

(105)
$$i\Omega D_0\big(\nabla_\perp\phi_0\boldsymbol{\cdot}\boldsymbol{n}_\perp\big) + i\varepsilon\Omega D_0\big(\nabla_\perp\phi_1\boldsymbol{\cdot}\boldsymbol{n}_\perp\big) = \varepsilon\frac{i\Omega D_0 p_0}{Z} + \varepsilon\widetilde{\mathcal{M}}\Big(\frac{D_0\boldsymbol{V}_0 p_0}{Z}\Big) + \mathcal{O}(\varepsilon^2),$$

where $\boldsymbol{V}_0 = U_0\boldsymbol{e}_x + \varepsilon\widetilde{\boldsymbol{V}}_{\perp 0}$.

9.2.1. $Z = \mathcal{O}(1)$. As before, we get to leading order

$$\nabla_\perp^2\phi_0 = 0, \quad\text{with}\quad \nabla_\perp\phi_0\boldsymbol{\cdot}\boldsymbol{n}_\perp = 0,$$

so $\phi_0 = \phi_0(X)$ and therefore $p_0 = p_0(X)$. To first order we have the same equation $\nabla_\perp^2\phi_1 = 0$ for ϕ_1, but the boundary condition is now

(106)
$$i\Omega D_0\big(\nabla_\perp\phi_1\boldsymbol{\cdot}\boldsymbol{n}_\perp\big) = \frac{i\Omega D_0 p_0}{Z} + \widetilde{\mathcal{M}}\Big(\frac{D_0\boldsymbol{V}_0 p_0}{Z}\Big).$$

In order to continue, we need from [16] the following property of the operator $\mathcal{M}$.

> For any sufficiently smooth vectorfield with $\boldsymbol{f}\boldsymbol{\cdot}\boldsymbol{n} = 0$ at $r = R$, we have
>
> $$\int_{\partial\mathcal{A}}\Big[\nabla\boldsymbol{\cdot}\boldsymbol{f} - \boldsymbol{n}\boldsymbol{\cdot}\big(\boldsymbol{n}\boldsymbol{\cdot}\nabla\boldsymbol{f}\big)\Big]\left\|\frac{\partial\boldsymbol{r}}{\partial x}\times\frac{\partial\boldsymbol{r}}{\partial\ell}\right\|\,\mathrm{d}\ell = \frac{\mathrm{d}}{\mathrm{d}x}\int_{\partial\mathcal{A}}(\boldsymbol{f}\times\boldsymbol{n})\boldsymbol{\cdot}\,\mathrm{d}\boldsymbol{\ell},$$
>
> where $(x,\ell)\mapsto\boldsymbol{r}(x,\ell)$ is a parameterization of the surface.

Since

$$\left\|\frac{\partial\boldsymbol{r}}{\partial x}\times\frac{\partial\boldsymbol{r}}{\partial\ell}\right\| = \sqrt{1 + \varepsilon^2\frac{R^2 R_X^2}{R^2 + R_\theta^2}} = 1 + \mathcal{O}(\varepsilon^2),$$

we have as a result

$$\int_{\partial\mathcal{A}}\widetilde{\mathcal{M}}\Big(\frac{D_0\boldsymbol{V}_0 p_0}{Z}\Big)\mathrm{d}\ell = \frac{\mathrm{d}}{\mathrm{d}X}\int_{\partial\mathcal{A}}\frac{D_0 U_0 p_0}{Z}\mathrm{d}\ell + \mathcal{O}(\varepsilon).$$

We apply this to the equation for ϕ_1, in order to obtain an equation for ϕ_0. From

$$\iint_\mathcal{A}\nabla_\perp^2\phi_1\,\mathrm{d}\sigma = \int_{\partial\mathcal{A}}\nabla_\perp\phi_1\boldsymbol{\cdot}\boldsymbol{n}_\perp\,\mathrm{d}\ell = 0,$$

together with (106) and noting that most functions depend on X only, it follows that

$$i\Omega D_0 p_0\mathcal{L} + \frac{\mathrm{d}}{\mathrm{d}X}\big(U_0 D_0 p_0\mathcal{L}\big) = 0, \quad\text{where}\quad \mathcal{L}(X) = \int_{\partial\mathcal{A}}\frac{1}{Z}\,\mathrm{d}\ell$$

($\mathcal{L}$ may be interpreted as the "total admittance" at X), with solution

$$p_0 = \text{constant }\frac{1}{U_0 D_0\mathcal{L}}\exp\left(-i\int^X\frac{\Omega}{U_0(\xi)}\,\mathrm{d}\xi\right).$$

Here ϕ_0 follows from (104) but is more difficult to obtain in explicit form. Note that this pressure field is not an acoustic wave, but it is of hydrodynamic nature. It does not propagate with the soundspeed, but with the mean flow velocity.

9.2.2. $Z = \mathcal{O}(\varepsilon)$. When $Z = \varepsilon Z_0$, we get for ϕ_0 the apparently difficult boundary condition

$$i\Omega D_0\big(\nabla_\perp \phi_0 \cdot \boldsymbol{n}_\perp\big) = \frac{i\Omega D_0 p_0}{Z_0} + \widetilde{\mathcal{M}}\Big(\frac{D_0 \boldsymbol{V}_0 p_0}{Z_0}\Big),$$

which is, analogous to the no-flow case, likely to be an eigenvalue problem with discrete eigenvalues Z_0 (apart from the trivial solutions $p_0 = 0$, $\phi_0 = \phi_0(X) \propto \exp(-i\Omega \int^X U_0(\xi)^{-1}\mathrm{d}\xi)$, i.e., hydrodynamically convected pressureless perturbations). If this conjecture is true, the possible eigenvalues vary with the geometry, and no other than the trivial solution is possible in a varying duct.

10. Conclusions. Generalizations of Webster's classic horn equation for nonuniform media, lined walls, and mean flow have been derived systematically, as an asymptotic perturbation problem for low Helmholtz number and slowly varying duct diameter. The conditions on frequency, acoustic medium, and duct geometry are explicitly indicated in terms of small parameter ε, the ratio between a typical length of duct variation and the duct diameter. The error and higher order corrections are also explicitly stated.

The presence of lining in a varying duct is shown to allow in general only trivial or merely hydrodynamic solutions. A curved duct is shown to produce the same equation if the radius of curvature is not smaller than the typical wavelength or duct length scale.

The approximation is nonuniform near a source or entrance. The prevailing boundary layer solution for an arbitrary duct cross section is given, together with the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ matching conditions to the outer ("Webster") region. From these expressions conditions are derived for which the $\mathcal{O}(\varepsilon)$-outer field is absent.

Appendix. On the spectrum of the Dirichlet-to-Neumann operator Ξ on smooth bounded domains in $\mathbb{R}^2$. We will show that the Dirichlet-to-Neumann operator Ξ, introduced in section 7 (see (80)), has a discrete spectrum of finite multiplicity. The basic idea is to relate the problem for the general simply connected open domain $\Omega \subset \mathbb{R}^2$ (which has apparently no explicit solution), via conformal mapping, to the corresponding problem for the unit disk D, which does have a simple explicit solution.

Note that the related result for an annular domain is entirely analogous.

Step 1. Consider the open unit-disk $D \subset \mathbb{R}^2$. Its boundary ∂D, the unit circle, is parametrized by the angle θ, with $0 \leq \theta < 2\pi$. The set of functions

$$\text{(A.1)} \qquad \mathrm{e}_n: \quad \theta \mapsto \mathrm{e}_n(\theta) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{in\theta}, \quad n \in \mathbb{Z},$$

establishes an orthonormal basis in $L_2(\partial D; \mathrm{d}\theta)$.[4] We introduce for real a the linear operator $\mathcal{N}_a$ in $L_2(\partial D, \mathrm{d}\theta)$, defined via the way it acts on the basis $\{\mathrm{e}_n\}$,

$$\text{(A.2)} \qquad \mathcal{N}_a: \quad \mathrm{e}_n \mapsto \mathcal{N}_a \mathrm{e}_n, \quad \text{with } \mathcal{N}_a \mathrm{e}_n(\theta) = (|n| + a)\mathrm{e}_n(\theta),$$

followed by linear extension and closure.

---

[4] $L_2(U; w(x)\mathrm{d}x)$ denotes the space of square integrable functions, defined on U, with inner product $\int_U \overline{f(x)}g(x)w(x)\,\mathrm{d}x$.

Let $u : \partial D \to \mathbb{C}$ be a sufficiently smooth function. Let $u_{\mathcal{H}}$, the harmonic extension of u, denote the (unique) solution of the Dirichlet problem

$$(\text{A.3}) \qquad \nabla^2 u_{\mathcal{H}}(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \in D, \text{ while } u_{\mathcal{H}}(\boldsymbol{x}) = u \text{ for } \boldsymbol{x} \in \partial D.$$

The normal derivative at the boundary ∂D produces a function

$$(\text{A.4}) \qquad\qquad\qquad \frac{\partial}{\partial n} u_{\mathcal{H}} : \ \partial D \to \mathbb{C}.$$

Altogether this defines the linear mapping $u \mapsto \frac{\partial}{\partial n} u_{\mathcal{H}}$, which is called the *Dirichlet-to-Neumann operator* in $L_2(\partial D; \mathrm{d}\theta)$. By noting that $\mathrm{e}_{n\mathcal{H}}(\boldsymbol{x}) = (x \pm \mathrm{i}y)^{|n|} = r^{|n|} \, \mathrm{e}^{\pm \mathrm{i}|n|\theta}$, and hence $\frac{\partial}{\partial n} \mathrm{e}_{n\mathcal{H}} = |n| \mathrm{e}_n$ at ∂D, it is easily verified that this operator is just equal to $\mathcal{N}_0$.

Step 2. Consider the bounded open domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. Let $v : \partial\Omega \to \mathbb{C}$ be a sufficiently smooth function. As in the previous section (just replace D by Ω), we introduce

$$(\text{A.5}) \qquad\qquad\qquad \Xi : \ v \mapsto \Xi v = \frac{\partial}{\partial n} v_{\mathcal{H}},$$

the Dirichlet-to-Neumann operator in $L_2(\partial\Omega; \mathrm{d}\theta)$. Thus $\Xi = \mathcal{N}_0$ if $\Omega = D$. We want to show that Ξ is nonnegative self-adjoint with a pure point spectrum of finite multiplicity. In the previous paragraph we showed this to be true in $L_2(\partial D; \mathrm{d}\theta)$.

The self-adjointness and nonnegativity follows, formally, from Green's first and second identities (see section 7). In order to achieve some spectral results, we invoke the Riemann mapping theorem and consider a conformal mapping $\beta : D \to \Omega$. The supposed smoothness of $\partial\Omega$ implies that the parametrization $\theta \mapsto \beta(\mathrm{e}^{\mathrm{i}\theta})$ for $\partial\Omega$ is such that both $|\beta'(\mathrm{e}^{\mathrm{i}\theta})|$ and its reciprocal are bounded.

Standard results from conformal mapping theory and harmonic functions on $\mathbb{R}^2$ lead to

$$(\text{A.6}) \qquad \Xi v\big(\beta(\mathrm{e}^{\mathrm{i}\theta})\big) = \left(\frac{\partial}{\partial n} v_{\mathcal{H}}\right)\big(\beta(\mathrm{e}^{\mathrm{i}\theta})\big) = \big|\beta'(\mathrm{e}^{\mathrm{i}\theta})\big|^{-1} \frac{\partial}{\partial n}(v \circ \beta)_{\mathcal{H}}(\mathrm{e}^{\mathrm{i}\theta}).$$

This means that, instead of the original problem, we could study the eigenvalue problem

$$(\text{A.7}) \qquad\qquad\qquad \mathcal{B}\mathcal{N}_0 u = \lambda u$$

in $L_2(\partial D, \mathrm{d}\theta)$, with $\mathcal{B}$ the multiplication operator defined by

$$(\text{A.8}) \qquad\qquad (\mathcal{B}w)(\theta) = B(\theta)w(\theta) = \big|\beta'(\mathrm{e}^{\mathrm{i}\theta})\big|^{-1} w(\theta).$$

(Although the inverse $\mathcal{B}^{-1}$ involves no more than division by the function $B(\theta)$, we retain for clarity the operator symbolism.)

Step 3. In order to turn the operator $\mathcal{B}\mathcal{N}_0$ into a self-adjoint one, we consider the eigenvalue problem in $L_2(\partial D; B^{-1}(\theta)\mathrm{d}\theta)$, which is topologically equivalent to $L_2(\partial D; \mathrm{d}\theta)$. Note that

$$(\text{A.9}) \qquad\qquad \big\{\theta \mapsto u(\theta)\big\} \mapsto \big\{\theta \mapsto B^{-\frac{1}{2}}(\theta)u(\theta)\big\}$$

furnishes a unitary transformation from $L_2(\partial D; B^{-1}(\theta)\mathrm{d}\theta)$ to $L_2(\partial D; \mathrm{d}\theta)$, because

$$(A.10) \qquad \int_0^{2\pi} \overline{u(\theta)}v(\theta)B^{-1}(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \overline{\left(B^{-\frac{1}{2}}(\theta)u(\theta)\right)}\left(B^{-\frac{1}{2}}(\theta)v(\theta)\right)\mathrm{d}\theta.$$

At the same time this implies that the eigenvalue problem (A.7) is unitary equivalent to the eigenvalue problem

$$(A.11) \qquad \mathcal{B}^{\frac{1}{2}}\mathcal{N}_0\mathcal{B}^{\frac{1}{2}}\varphi = \lambda\varphi,$$

with $\varphi = \mathcal{B}^{-\frac{1}{2}}u$.

Step 4. If we can show that $(\mathcal{I}+\mathcal{B}^{\frac{1}{2}}\mathcal{N}_0\mathcal{B}^{\frac{1}{2}})^{-1}$ (where $\mathcal{I}$ is the identity) is a compact self-adjoint operator, we are ready. In that case it has a discrete spectrum with finite multiplicity [31], and the same holds, a fortiori, for $\mathcal{B}^{\frac{1}{2}}\mathcal{N}_0\mathcal{B}^{\frac{1}{2}}$.

Take a positive and sufficiently small such that $\theta \mapsto B^{-1}(\theta) - a$ is still positive and uniformly bounded away from zero. By noting that $\mathcal{N}_a = \mathcal{N}_0 + a\mathcal{I}$, we can rewrite

$$(A.12) \qquad \mathcal{I} + \mathcal{B}^{\frac{1}{2}}\mathcal{N}_0\mathcal{B}^{\frac{1}{2}} = \mathcal{B}^{\frac{1}{2}}\mathcal{N}_a^{\frac{1}{2}}\left\{\mathcal{N}_a^{-\frac{1}{2}}(\mathcal{B}^{-1} - a\mathcal{I})\mathcal{N}_a^{-\frac{1}{2}} + \mathcal{I}\right\}\mathcal{N}_a^{\frac{1}{2}}\mathcal{B}^{\frac{1}{2}}.$$

The operator between brackets, { }, is bounded, positive, and self-adjoint and has an inverse with the same properties. We thus find

$$(A.13) \qquad \left(\mathcal{I} + \mathcal{B}^{\frac{1}{2}}\mathcal{N}_0\mathcal{B}^{\frac{1}{2}}\right)^{-1} = \mathcal{B}^{-\frac{1}{2}}\mathcal{N}_a^{-\frac{1}{2}}\left\{\mathcal{N}_a^{-\frac{1}{2}}(\mathcal{B}^{-1} - a\mathcal{I})\mathcal{N}_a^{-\frac{1}{2}} + \mathcal{I}\right\}^{-1}\mathcal{N}_a^{-\frac{1}{2}}\mathcal{B}^{-\frac{1}{2}},$$

which is a composition of operators. Since the factor $\mathcal{N}_a^{-\frac{1}{2}}$ is compact, also $(\mathcal{I} + \mathcal{B}^{\frac{1}{2}}\mathcal{N}_0\mathcal{B}^{\frac{1}{2}})^{-1}$ is a compact operator.

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