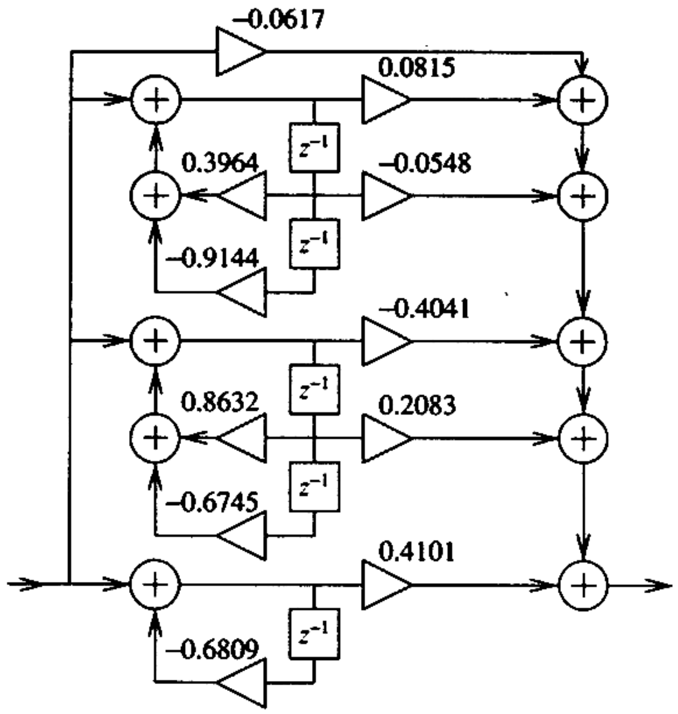

Figure 9.45 Parallel realization of fifth-order elliptic IIR example filter.

relatively small pole magnitudes, and considerable computation savings recommend this class of half-band filters for multirate applications.

9.8 HILBERT TRANSFORMER

The Hilbert transform is a set of mathematical equations relating the real and imaginary parts, or the magnitude and phase, of the Fourier transform of certain signals. Based on the properties of the discrete Hilbert transform, an *ideal Hilbert transformer* is defined as a DTLTI system with the frequency response

$$H_{HT}(e^{j2\pi f}) = \begin{cases} -j, & 0 \le f < \frac{1}{2} \\ +j, & -\frac{1}{2} \le f < 0 \end{cases}$$

An ideal Hilbert transformer is also called a *90-degree phase-shifter*. It is non-causal and cannot be realized. Any realization that approximates the ideal Hilbert transformer is referred to as a *Hilbert transformer*. In this section we present a realization of Hilbert transformers using half-band filters.

9.8.1 Half-Band Filter and Hilbert Transformer

A half-band filter is defined by the passband-stopband symmetry related to $f = \frac{1}{4}$. If $H(z)$ is the transfer function of the half-band filter, then $H_H(z) = H(z/e^{j\pi/2}) =$

$H(-jz)$ is the transfer function of a complex filter with passband-stopband symmetry related to $f = \frac{1}{2}$. The squared magnitude responses of the Hilbert transformer and the half-band filter are plotted in Fig. 9.46. If the input sequence to a filter with $H_H(z)$ is real, say $x(n)$, then the output sequence is complex, say $y(n) = y_r(n) + jy_i(n)$, where $y_r(n)$ and $y_i(n)$ are real. The signal $y(n)$ is called a *complex analytic signal.* A filter with one input and two outputs, one for $y_r(n)$ and one $y_i(n)$, is the Hilbert transformer. The real and imaginary outputs of the complex filter form a Hilbert transform pair over the specified frequency range [88].

The key properties of the elliptic half-band IIR filter are

$$\begin{aligned} F_p &= \frac{1}{2} - F_s \\ \left|H(e^{j\pi/2})\right|^2 &= \frac{1}{2} \\ \Delta_p = \Delta_s \quad &\Leftrightarrow \quad a_p = 10\log_{10}\left(1 + \frac{1}{10^{a_s/10} - 1}\right) \end{aligned} \tag{9.85}$$

where

$$\begin{aligned} \Delta_p &= 1 - |H(e^{j2\pi F_p})|^2 \\ \Delta_s &= |H(e^{j2\pi F_s})|^2 \end{aligned} \tag{9.86}$$

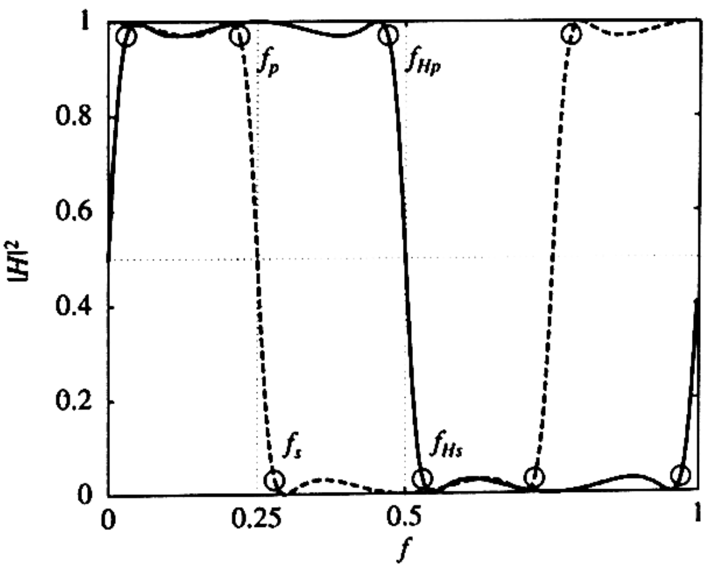

Figure 9.46 Squared magnitude response of Hilbert transformer (solid line) and half-band filter (dashed line).

The key properties of the Hilbert transformer are

$$
\begin{aligned}
F_{Hp} &= 1 - F_{Hs} \\
|H_H(e^{j\pi})|^2 &= \frac{1}{2} \\
\Delta_{Hp} = \Delta_{Hs} \quad &\Leftrightarrow \quad a_{Hp} = 10 \log_{10}\left(1 + \frac{1}{10^{a_{Hs}/10} - 1}\right)
\end{aligned}
\tag{9.87}
$$

where

$$
\begin{aligned}
\Delta_{Hp} &= 1 - |H_H(e^{j2\pi F_{Hp}})|^2 \\
\Delta_{Hs} &= |H_H(e^{j2\pi F_{Hs}})|^2
\end{aligned}
\tag{9.88}
$$

Hilbert transformers can be designed from half-band filters in a straightforward manner, by replacing z with $-jz$.

The transfer function poles of the half-band IIR filter are on the imaginary axis of the z-plane. The realization of the filter is based on the sum of two allpass transfer functions, $H_a(z)$ and $H_b(z)$ (Fig. 9.47):

$$H(z) = \frac{1}{2}(H_a(z) + H_b(z)) \tag{9.89}$$

with

$$H_a(z) = \prod_{i=[(n+7)/4]}^{(n+1)/2} \frac{\beta_i + z^{-2}}{1 + \beta_i z^{-2}} \tag{9.90}$$

$$H_b(z) = z^{-1} \prod_{i=3}^{[(n+1)/4]} \frac{\beta_i + z^{-2}}{1 + \beta_i z^{-2}} \tag{9.91}$$

where β_i is the square magnitude of a pole z_i, such that

$$0 \le \beta_i = |z_i|^2 < 1 \tag{9.92}$$

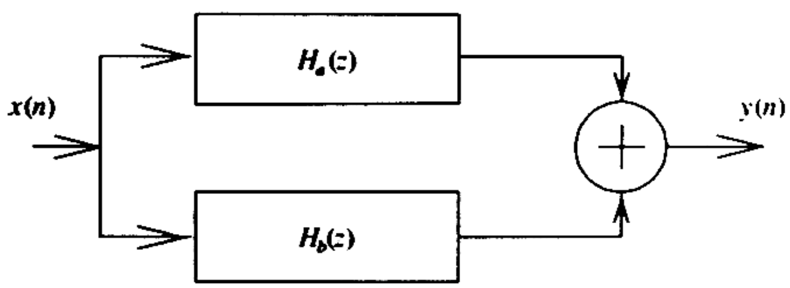

Figure 9.47 Realization of a half-band filter.

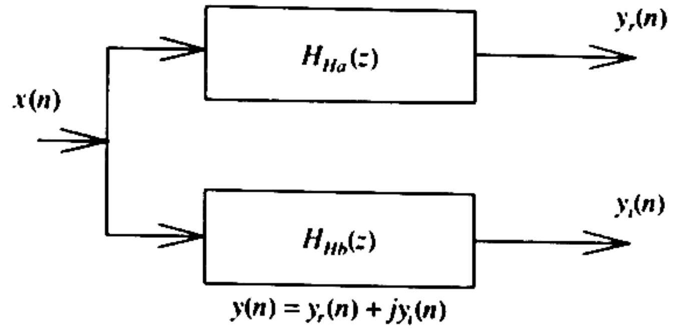

Figure 9.48 Realization of Hilbert transformer (complex filter).

The poles of the Hilbert transformer are on the real axis of the z plane. Replacing z with $-jz$ in $H_a(z)$ and $H_b(z)$, we obtain the transfer function of the complex filter (Fig. 9.48):

$$H_H(z) = H_{Ha}(z) + jH_{Hb}(z) \tag{9.93}$$

$$H_{Ha}(z) = \prod_{i=[(n+7)/4]}^{(n+1)/2} \frac{\beta_i - z^{-2}}{1 - \beta_i z^{-2}} \tag{9.94}$$

$$H_{Hb}(z) = z^{-1} \prod_{i=3}^{[(n+1)/4]} \frac{\beta_i - z^{-2}}{1 - \beta_i z^{-2}} \tag{9.95}$$

The allpass transfer functions $H_{Ha}(z)$ and $H_{Hb}(z)$ correspond to branches of a 90-degree phase-shifter (also called the $\frac{\pi}{2}$ *phase splitter*) whose outputs approximate the Hilbert transform pair [89].

We can rewrite Eqs. (9.94) and (9.95) as follows:

$$H_{Ha}(z) = \prod_{i=[(n+7)/4]}^{(n+1)/2} \frac{(-\beta_i) + z^{-2}}{1 + (-\beta_i)z^{-2}} \tag{9.96}$$

$$H_{Hb}(z) = (-1)^{(n+1)/2}\, z^{-1} \prod_{i=3}^{[(n+1)/4]} \frac{(-\beta_i) + z^{-2}}{1 + (-\beta_i)z^{-2}} \tag{9.97}$$

Eqations (9.96) and (9.97) show that the coefficients of the Hilbert transformer are the negative coefficients of the half-band filter. Therefore, the procedure for designing multiplierless elliptic IIR filters can be used for the design of multiplierless Hilbert transformers.

9.8.2 Design of Hilbert Transformer

The Hilbert transformer can be specified by the lower stopband edge frequency, F_{Hs}, and the minimum stopband attenuation, A_s, that is, $S_H = \{F_{Hs}, A_s\}$. The corresponding half-band filter has the specification $S = \{F_s, A_s\}$ with $F_s = F_{Hs} - \frac{1}{4}$. Next, the coefficients of the half-band filter are calculated to meet $S = \{F_s, A_s\}$. If the coefficients of the half-band filter are β_i, then the coefficients of the Hilbert transformer are equal to $-\beta_i$.

Example 9.2

Assume a half-band filter specification $F_s = 0.28$, along with $A_s = 46$ dB and $S = \{F_s, A_s\} = \{0.28, 46\}$. For $n = 9$ the two cases exist:

1. $a_s = A_s$, $f_s = 0.275 < F_s$;
2. $A_s < a_s = 57.2$ and $f_s = F_s$.

The specification is satisfied if $46 \le a_s \le 57.2$ and $0.275 \le f_s \le 0.28$.

For the two cases we find:

1. $n = 9$, $a_s = A_s = 46$ dB, $a_p = 0.00011$, $\xi = 1.217$, $f_s = 0.2656$, $f_p = 0.2344$ and $\beta_2 = 0.1532$, $\beta_3 = 0.7336$, $\beta_4 = 0.4646$, $\beta_5 = 0.9202$;
2. $n = 9$, $a_s = 57.18$ dB, $a_s = 0.0000083$, $\xi = 1.461$, $f_s = 0.28$, $f_p = 0.2216$, and $\beta_2 = 0.1091$, $\beta_3 = 0.3616$, $\beta_4 = 0.6335$, $\beta_5 = 0.8774$. ◆

Figs. 9.49 and 9.50 show attenuation: solid lines for $\beta_2 = 0.1206$, $\beta_3 = 0.6628$, $\beta_4 = 0.3900$, $\beta_5 = \beta_{max} = 1 - 1/2^3 + 1/2^6$, dashed lines for quantized coefficients $\beta_{2q} = 1/2^3 - 1/2^8$, $\beta_{3q} = 1/2 + 1/2^3 + 1/2^5 + 1/2^7 = (1 + 1/2^2)(1/2 + 1/2^5)$, $\beta_{4q} = 1/2^2 + 1/2^3 + 1/2^6$, $\beta_5 = \beta_{max} = 1 - 1/2^3 + 1/2^6$.

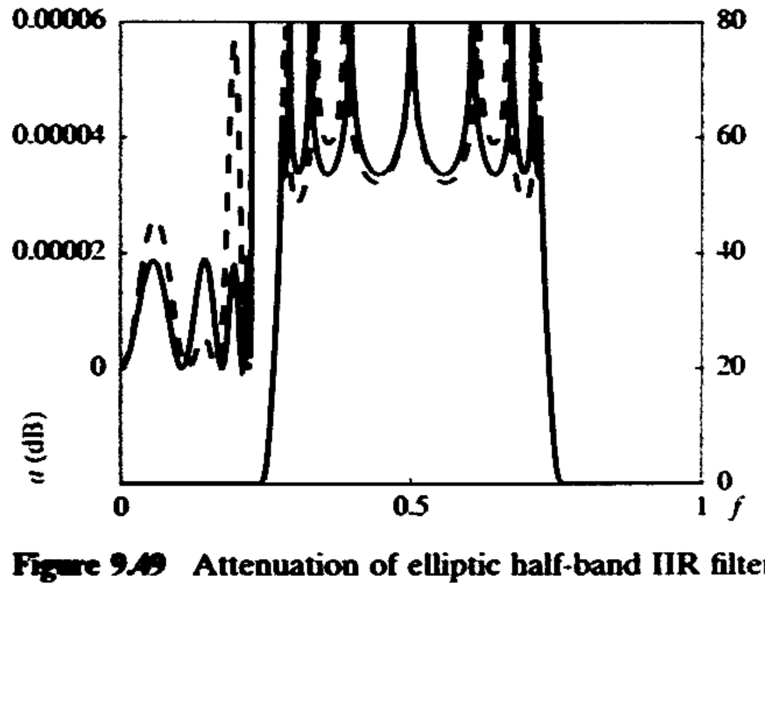

Figure 9.49 Attenuation of elliptic half-band IIR filter.

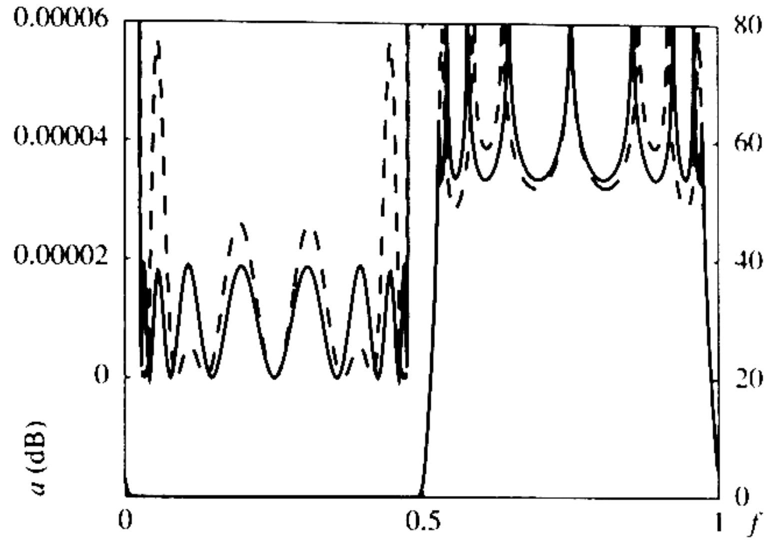

Figure 9.50 Attenuation of IIR Hilbert transformer.

A realization of the ninth-order half-band filter is shown in Fig. 9.51, and the corresponding complex filter is presented in Fig. 9.52, for $f_s = 0.2751$ and $a_s = 53.64$ dB. The corresponding magnitude responses are shown in Figs. 9.49 and 9.50.

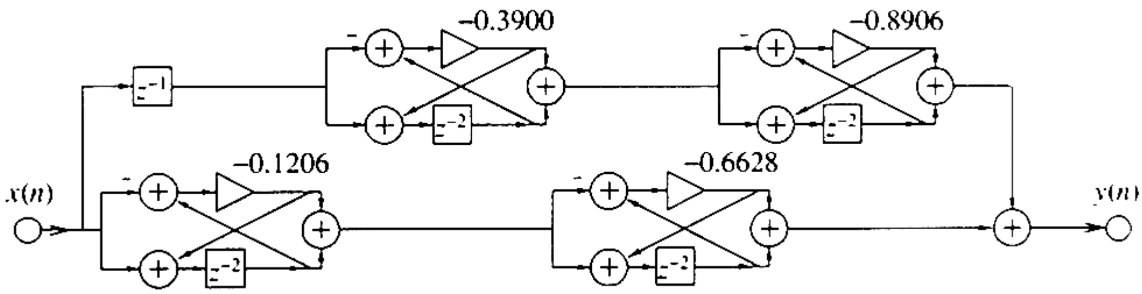

Figure 9.51 Ninth-order elliptic half-band IIR filter.

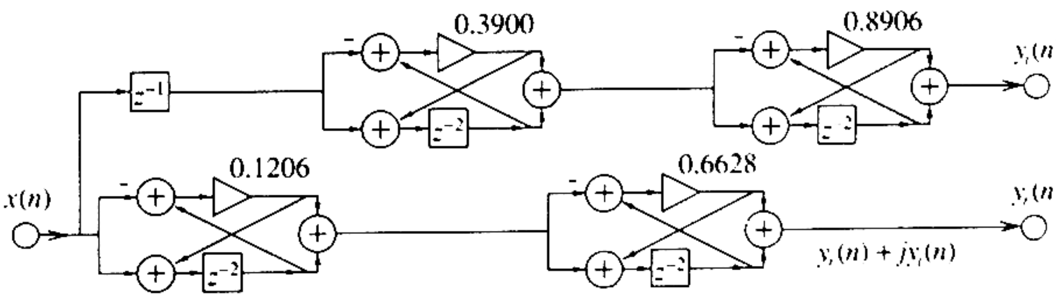

Figure 9.52 Realization of Hilbert transformer by using ninth-order elliptic complex IIR filter.

9.9 MULTIPLIERLESS ELLIPTIC IIR FILTERS

An analytical expression for amplitude response sensitivity is derived for the filter realizations consisting of two allpass subfilters in parallel. It is shown that the amplitude