DESIGN OF MULTIPLIERLESS ELLIPTIC IIR HALFBAND FILTERS AND HILBERT TRANSFORMERS

Miroslav D. Lutovac[1] and *Ljiljana D. Milić*[2]
[1] IRITEL Institute, Batajnički put 23,
11080 Belgrade, Yugoslavia, elutovac@ubbg.etf.bg.ac.yu
[2] Mihajlo Pupin Institute, Volgina 15,
11000 Belgrade, Yugoslavia, emilicl@ubbg.etf.bg.ac.yu

ABSTRACT

A new design method for multiplierless IIR halfband filters and Hilbert transformers is presented. The coefficients of the halfband filter and the Hilbert transformer are determined using the same procedure. The minimal number of shift-and-add operation design (*multiplierless*) is based on the sensitivity analysis and the phase tolerance scheme.

1 INTRODUCTION

In this paper we present a new design method for multiplierless IIR halfband filters and Hilbert transformers. "Multiplierless" means that all multiplication constants are implemented with few shifters and adders. It has been shown in [1] and [2] that the transfer function of an IIR halfband filter can be developed from an elliptic minimal Q-factor analog prototype. The multiplierless elliptic IIR filter design for the structures based on the parallel connection of two allpass networks has been presented on [3].

In this paper a new generalized approach for halfband filters and Hilbert transformers is given. This paper examines halfband filters and Hilbert transformers and gives new general analytical expressions for amplitude and phase sensitivities. Using this sensitivity analysis the new design technique based on the phase tolerance scheme is presented that may be applied equally for a halfband filter or Hilbert transformer specifications. Although the sensitivity of an IIR filter is high, very sharp multiplierless filters can be obtained by the choice of the transfer function parameters.

2 TRANSFER FUNCTION AND REALIZATION STRUCTURE

A halfband filter is defined by the passband-stopband symmetry related to $\omega=\pi/2$. If $H(z)$ is the transfer function of the halfband filter, $H(z/e^{j\pi/2})$ is the transfer function of the complex filter with passband-stopband symmetry related to $\omega=\pi$. This follows from the modulation property of z transform. The real and imaginary outputs of the complex filter form a Hilbert transform pair over the specified frequency range [4].

It has been shown in [1] and [2] that an elliptic minimal Q-factors filter is an analog prototype of the IIR halfband filter. Due to the modulation property of the z-transform, the elliptic minimal Q-factor filter can be considered as an analog prototype for Hilbert transformer, as well. Therefore, instead of developing a special computer program for IIR Hilbert transformers based on elliptic functions, it is straightforward to use existing programs for digital filters and then replace z with $z/e^{j\pi/2}$. For example *ellip.m* from MATLAB can be used.

The general property of this class of filters is that the passband ripple a_p and the stopband attenuation a_a are related by the following relation:

$$a_p = 10\log_{10}\left(1+\frac{1}{10^{a_a/10}-1}\right) \tag{1}$$

The poles of the halfband filter are placed on the imaginary axis, and the poles of the Hilbert transformer are rotated for $\pi/2$, i.e. they are placed on the real axis of the z plane. The realization is based on the sum of two allpass functions $H_a(z)$ and $H_b(z)$ gives:

$$H(z) = \frac{1}{2}\left(H_a(z)+H_b(z)\right) \tag{2}$$

For the halfband filter $H_a(z)$ and $H_b(z)$ are :

$$H_a(z) = \prod_{i=[(n+7)/4]}^{(n+1)/2} \frac{\beta_i + z^{-2}}{1+\beta_i z^{-2}} \tag{3}$$

$$H_b(z) = z^{-1}\prod_{i=3}^{[(n+1)/4]} \frac{\beta_i + z^{-2}}{1+\beta_i z^{-2}} \tag{4}$$

where β_i is the square module of the pole z_i. Replacing z with $z/e^{j\pi/2}$ in $H_a(z)$ and $H_b(z)$, we obtain the complex filter:

$$H_{HT}(z) = H_{HTa}(z) + jH_{HTb}(z) \tag{5}$$

$$H_{HTa}(z) = \prod_{i=[(n+7)/4]}^{(n+1)/2} \frac{\beta_i - z^{-2}}{1-\beta_i z^{-2}} \tag{6}$$

$$H_{HTb}(z) = z^{-1} \prod_{i=3}^{[(n+1)/4]} \frac{\beta_i - z^{-2}}{1 - \beta_i z^{-2}} \tag{7}$$

$H_{HTa}(z)$ and $H_{HTb}(z)$ present allpass branches of a $\pi/2$ phase splitter whose outputs approximate the Hilbert transform pair [5].

We can rewrite Eqs. (6) and (7)

$$H_{HTa}(z) = \prod_{i=[(n+7)/4]}^{(n+1)/2} \frac{(-\beta_i) + z^{-2}}{1 + (-\beta_i) z^{-2}} \tag{8}$$

$$H_{HTb}(z) = (-1)^{(n+1)/2}\, z^{-1} \prod_{i=3}^{[(n+1)/4]} \frac{(-\beta_i) + z^{-2}}{1 + (-\beta_i) z^{-2}} \tag{9}$$

Eqs. (8) and (9) show that the coefficients of the Hilbert transformer are the negative coefficients of the halfband filter. This implies that the procedure derived for multiplierless elliptic IIR filter can be used for the design of the multiplierless Hilbert transformer.

3 TRANSFER FUNCTION DESIGN

The amplitude response $\mathcal{A}(\omega)$ for the realizations based on Eqs. (2) and (5) can be expressed by the phase difference function $\Psi(\omega)$:

$$\mathcal{A}(\omega) = \cos(\Psi(\omega)) \tag{10}$$

where $\Psi(\omega)$ is the phase difference between $H_a(z)$ and $H_b(z)$:

$$\Psi(\omega) = \frac{\varphi_a(\omega) - \varphi_b(\omega)}{2} = \frac{1}{2} \sum_{i=2}^{(n+1)/2} \pm\varphi_i(\omega) \tag{11}$$

For $\Psi(\omega) = \pi/2$ we have $\mathcal{A}(\omega) = 0$. For the required minimal stopband attenuation A_a given in dB, we compute the permitted phase tolerance, $\Psi(\omega) - \pi/2$, in the stopband which is denoted by D_a:

$$D_a = \left| \tfrac{\pi}{2} - \cos^{-1}(10^{(-A_a/20)}) \right| \tag{12}$$

Fig. 1 (solid line) illustrates a typical behavior of the function $\Psi(\omega) - \pi/2$ for the elliptic halfband filter for $n = 9$, and $f_a = 0.28$. The permitted phase tolerance $\pm D_a$ is calculated for $A_a = 46$ dB. Fig. 2 (solid line) is for the corresponding Hilbert transformer whose rejection band is defined over $0.53 < f < 0.97$. This way, the design of the halfband filter and the Hilbert transformer are regarded as a phase approximation problem. Since the Hilbert transformer is usually considered as a phase splitting network, the proposed approach is particularly convenient.

The design starts from a specification $S = \{F_a, D_a\}$ or $S = \{F_a, A_a\}$, where F_a is the stopband edge frequency of the halfband filter. First, we determine the filter order n (n is odd integer). Next, if D_a is specified, we calculate A_a from Eq. (12)

$$A_a = -20\log_{10}(\cos(\frac{\pi}{2} - D_a)) \tag{13}$$

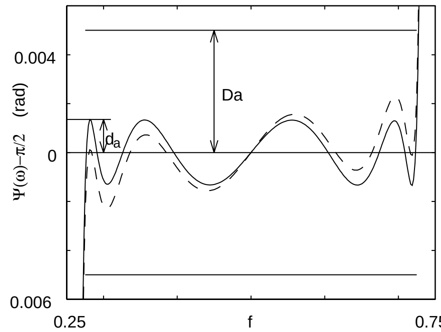

Figure 1: Halfband filter. Phase tolerance scheme.

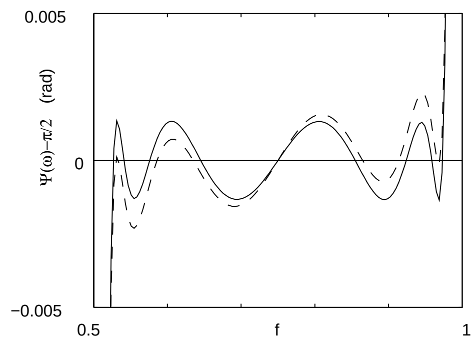

Figure 2: Hilbert transformer. Phase tolerance scheme.

The axillary variable L is found for known A_a

$$L = 10^{A_a/10} - 1 \tag{14}$$

The maximal passband attenuation, a_p, depends on A_a or L only

$$a_p = 10\log_{10}(1 + \frac{1}{L}) \tag{15}$$

We use upper-case letters for the specification quantities $(D_a, A_a, A_p, F_a, F_p)$ and lower-case letters for actual values $(d_a, a_a, a_p, f_a, f_p)$.

The passband and stopband edge frequencies are functions of A_a only, and can be calculated using the approximate procedure adapted from [6]

$$t = \frac{1}{2} \frac{1 - \sqrt[4]{1 - \frac{1}{L^2}}}{1 + \sqrt[4]{1 - \frac{1}{L^2}}} \tag{16}$$

$$q = t + 2t^5 + 15t^9 + 150t^{13} \tag{17}$$

$$g = e^{log(q)/n} \tag{18}$$

$$q_0 = \frac{g + g^9 + g^{25} + g^{49} + g^{81} + g^{121} + g^{169}}{1 + 2(g^4 + g^{16} + g^{36} + g^{64} + g^{100} + g^{144})} \tag{19}$$

$$\Omega_a = \frac{1}{\sqrt{1 - \left(\frac{1-2q_0}{1+2q_0}\right)^4}} \tag{20}$$

Finally, the edge frequencies for the halfband filter are

$$f_a = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\frac{1}{\sqrt{\Omega_a}} \tag{21}$$

$$f_p = \frac{1}{\pi}\tan^{-1}\frac{1}{\sqrt{\Omega_a}} \tag{22}$$

The calculated edge frequency cannot be larger than specified: $f_a \leq F_a$. For $f_a > F_a$ we increase the filter order and repeat the procedure.

Alternatively, the design can start from specification $S = \{F_a, D_a\}$ and $f_a = F_a$. For that case, the design procedure is different; first we calculate f_p

$$f_p = \frac{1}{2} - F_a, \quad f_a = F_a \tag{23}$$

The selectivity factor of analog filter prototype is

$$\Omega_a = \frac{1}{\tan^2(\frac{\pi}{2} - \pi f_a)} \tag{24}$$

and a_a, a_p and d_a can be calculated using the procedure

$$\Omega_a \geq \sqrt{2} \quad \rightarrow \quad \begin{cases} t = \frac{1}{2}\frac{1-\sqrt[4]{1-\frac{1}{\Omega_a^2}}}{1+\sqrt[4]{1-\frac{1}{\Omega_a^2}}} \\ q = t + 2t^5 + 15t^9 + 150t^{13} \end{cases}$$

$$\Omega_a < \sqrt{2} \quad \rightarrow \quad \begin{cases} t = \frac{1}{2}\frac{1-\frac{1}{\sqrt{\Omega_a}}}{1+\frac{1}{\sqrt{\Omega_a}}} \\ q_p = t + 2t^5 + 15t^9 + 150t^{13} \\ q = e^{\pi^2/\log(q_p)} \end{cases} \tag{25}$$

$$L = \frac{1}{4}\sqrt{\frac{1}{q^n} - 1} \tag{26}$$

$$a_p = 10\log_{10}(1 + \frac{1}{L}) \tag{27}$$

$$a_a = 10\log_{10}(1 + L) \tag{28}$$

$$d_a = \left|\pi/2 - \cos^{-1}(10^{-a_a/20})\right| \tag{29}$$

The calculated phase deviation cannot be larger than specified: $d_a \leq D_a$. For $d_a > D_a$ we have to choose the larger filter order n and repeat the procedure.

Finally, we can use any standard program for calculating the transfer function poles; for example in MATLAB: $[z, p, k] = ellip(n, a_p, a_a, 2f_p)$. Since all poles are on the imaginary axis, the filter coefficients of the second order sections are $\beta_i = |p_i|^2$.

Let us consider the specification: $F_a = 0.28$ and $D_a = 0.005$ rad, or $A_a = 46$ dB. For $n = 9$ the two cases are: a) $d_a = D_a$, $a_a = A_a$, $f_a = 0.275 < F_a$ and

b) $d_a = 0.00138 < D_a$, $A_a < a_a = 57.2$ and $f_a = F_a$. The specification is satisfied for: $0.00138 \leq d_a \leq 0.005$, or $46 \leq a_a \leq 57.2$, or $0.275 \leq f_a \leq 0.28$.

For those two cases we find:

a) $n = 9$, $d_a = D_a = 0.005$ rad, $a_a = A_a = 46$ dB, $a_p = 0.00011$, $\Omega_a = 1.217$, $f_a = 0.2656$, $f_p = 0.2344$ and $\beta_2 = 0.1532$, $\beta_3 = 0.7336$, $\beta_4 = 0.4646$, $\beta_5 = 0.9202$.

b) $n = 9$, $d_a = 0.001384$ rad, $a_a = 57.18$ dB, $a_p = 0.0000083$, $\Omega_a = 1.461$, $f_a = 0.28.$, $f_p = 0.2216$, and $\beta_2 = 0.1091$, $\beta_3 = 0.3616$, $\beta_4 = 0.6335$, $\beta_5 = 0.8774$, and the permited ranges for quantities are $0.0014 \leq d_a \leq 0.005$, $46 \leq a_a \leq 57$, $0.00001 \leq a_p \leq 0.0001$, $0.2656 \leq f_a \leq 0.28$, $0.2216 \leq f_p \leq 0.2344$ and $0.109 \leq \beta_2 \leq 0.153$, $0.634 \leq \beta_4 \leq 0.734$, $0.362 \leq \beta_3 \leq 0.465$, $0.877 \leq \beta_5 \leq 0.92$.

4 SENSITIVITY ANALYSIS

The amplitude response sensitivity of digital filters realized by a parallel combination of two allpass networks is considered in [3]. It is shown that the amplitude response sensitivity to some constant β_i can be expressed as a product of the filter reflectance function and the phase sensitivity of the section which implements β_i. In this paper, we present new expressions for the phase sensitivity for halfband filters. If $\varphi_i(\omega)$ is the phase response of the ith-section, and $\partial\varphi_i/\partial\beta_i$ is the corresponding phase sensitivity, then:

$$\varphi_i(\omega) = \tan^{-1}\frac{-(1-\beta_i^2)\sin 2\omega}{2\beta_i - (1+\beta_i^2)\cos 2\omega} \tag{30}$$

$$\frac{\partial\varphi_i(\omega)}{\partial\beta_i} = \frac{2\sin 2\omega}{1+\beta_i^2+2\beta_i\cos 2\omega} \tag{31}$$

The extreme values of the function $\frac{\partial\varphi_i}{\partial\beta_i}$ are:

$$\max\left|\frac{\partial\varphi_i(\omega)}{\partial\beta_i}\right| = \frac{2}{1-\beta_i^2} \tag{32}$$

and they occur at the frequencies:

$$\omega_1 = \frac{1}{2}\cos^{-1}\left(\frac{-2\beta_i}{1-\beta_i^2}\right) \tag{33}$$

$$\omega_2 = \pi - \frac{1}{2}\cos^{-1}\left(\frac{-2\beta_i}{1-\beta_i^2}\right) \tag{34}$$

To obtain the sensitivity functions for Hilbert transformers, we have to shift the sensitivity functions of the halfband filter for $\pi/2$.

5 MULTIPLIERLESS DESIGN

The aim of our design approach is to replace constants β_i with the new values β_{iq} chosen to be implemented with as small number of shifters and adders as possible. This is achieved using a design margin and the sensitivity functions. Since the passband sensitivity is very low, only the stopband has to be considered.

The allpass section implementing the pole pair closest to the unit circle has the highest sensitivity, Eq. (32). Therefore, the quantization of the largest pole, $\max(\beta_i)$, can significantly degenerate the amplitude and phase responses. Since the largest pole can be chosen from a range, (in the previous example $0.877 \leq \beta_5 \leq 0.92$) we have to find $d_a \leq D_a$, $a_a \geq A_a$, or $f_a \leq F_a$, so that the quantization error is 0.

The quantization of remaining less sensitive constants is performed by the computation of the range of permitted values. For each β_i, the new value β_{iq} is chosen from the range according to the minimal number of shift-and add operations.

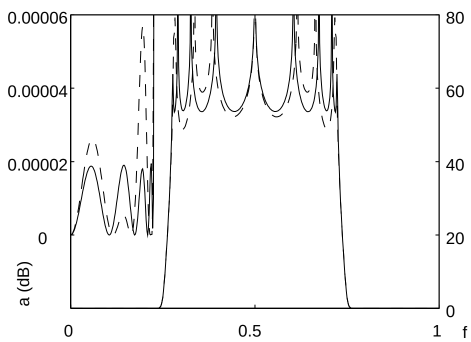

Figure 3: Halfband filter. Attenuation characteristics.

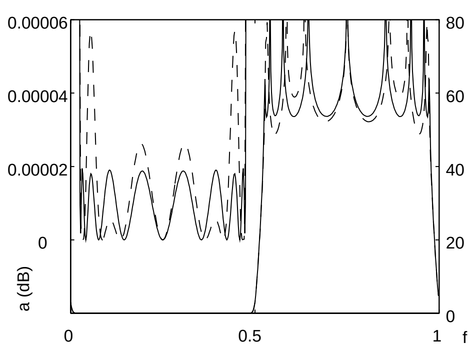

Figure 4: Hilbert transformer. Attenuation characteristics of the complex filter.

6 DESIGN OF HILBERT TRANSFORMER

The Hilbert transformer can be specified by the lowest edge frequency and the phase tolerance or the stopband attenuation: $S_{HT} = \{F_{HTa}, D_a\}$ or $S_{HT} = \{F_{HTa}, A_a\}$. The corresponding edge frequency of the halfband filter is $F_a = F_{HTa} - 0.25$. The filter coefficients are calculated for the halfband filter specification $S = \{F_a, D_a\}$ or $S = \{F_a, A_a\}$. Next, d_a, a_a, or f_a is found so that the quantization error of the largest pole, $\max(\beta_i)$, is 0. The coefficients of the Hilbert transformer are equal to $-\beta_i$, where β_i are the coefficients of the halfband filter.

7 EXAMPLE

Figs. 1 and 2 (dashed lines) display the phase differences of the multiplierless design for the example of 9-th order halfband filter and Hilbert transformer. Figs. 3 and 4 display corresponding attenuation characteristics: solid lines for $\beta_2 = 0.1206$, $\beta_3 = 0.6628$, $\beta_4 = 0.3900$, $\beta_5 = \beta_{\max} = 1 - 1/2^3 + 1/2^6$, dashed lines for $\beta_{2q} = 1/2^3 - 1/2^8$, $\beta_{3q} = 1/2 + 1/2^3 + 1/2^5 + 1/2^7 = (1 + 1/2^2)(1/2 + 1/2^5)$, $\beta_{4q} = 1/2^2 + 1/2^3 + 1/2^6$, $\beta_5 = \beta_{\max} = 1 - 1/2^3 + 1/2^6$. For lower order and less selective filters the solutions are even simpler.

8 CONCLUSION

A straightforward procedure is developed for the design of multiplierless halfband elliptic IIR filters and Hilbert transformers. The efficiency of the procedure is demonstrated by example of a selective filter that is very sensitive to the coefficient quantization. The quantization error can be minimized using simple closed-form sensitivity relations. Thus the halfband filter and the Hilbert transformer can be realized with minimal number of shift-and-add operations.

9 ACKNOWLEDGMENT

The authors would like to thank to Dr. Charles M. Rader for his valuable comments, encouragement and discussions on the subject.

References

[1] L. D. Milić and M. D. Lutovac, "Reducing the number of multipliers in the parallel realization of halfband elliptic IIR filters," *IEEE Trans. Signal Processing*, vol. 44, no. 10, pp. 2619–2623, Oct. 1996.

[2] M. D. Lutovac and L. D. Milić, "Design of computationally efficient elliptic IIR filters with a reduced number of shift-and-add operations in multipliers," *IEEE Trans. Signal Processing*, vol. 45, no. 10, pp. 2422–2430, Oct. 1997.

[3] Lj. Milić and M. D. Lutovac, "Design of multiplierless IIR filters," in *IEEE Int. Conf. Acoustics, Speech, Signal Processing, ICASSP97*, Munich, Germany, Apr. 1997, pp. 2201–2204.

[4] P. A. Regalia, "Special filter design," in S.K.Mitra and J.F.Kaiser, *Handbook of Digital Signal Processing*, John Wiley, 1993, pp. 907–980.

[5] C. M. Rader, "A simple method for sampling in-phase and quadrature components," *IEEE Trans. Aerospace Electronic systems*, vol. 20, no. 6, pp. 821–824, Nov. 1984.

[6] H. J. Orchard, "Adjusting the parameters in elliptic–function filters," *IEEE Trans. on CAS*, vol. 37, pp. 631–633, May 1990.